EUCLID

Taken from a Brass Coin in the Repository of the late Queen Christian of Sweden.

Euclid *the Mathematician was of* Alexandria *where he Taught in the Reign of* Ptolomy Lagus *in the* CXX Olympiad *and Year of* Rome 454. *He wrote many things relating to Musick and Geometry; But his* XV *Books of Elements (of which he is generaly thought to be only the Collector) are most applauded. the two last are attributed to* Hypsicles *of* Alexandria *and not to him.* *Cardan Vossius.*

ELEMENTS of EUCLID:

WITH

SELECT THEOREMS

OUT OF

ARCHIMEDES.

By the Learned ANDREW TACQUET.

To which are added,

PRACTICAL COROLLARIES, shewing the Uses of many of the Propositions.

By WILLIAM WHISTON, M. A.

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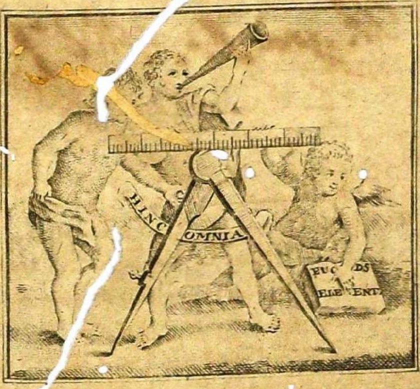
The ELEVENTH EDITION.

With an APPENDIX of PRACTICAL GEOMETRY, and Forty New Figures, with a Brief, Independent Demonstration of certain Select and most useful Propositions.

By S. [illegible].



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AN

HISTORICAL ACCOUNT

OF THE

RISE AND PROGRESS

OF THE

MATHEMATICKS.

IT ſeemed meet to me, when I was about to ſet forth the elements of the *Mathematicks* to premiſe a few things concerning the riſe and excellency of this ſcience, that its candidates may underſtand what a kind of ſcience it is, to which they are about to dedicate themſelves; and that it may be made manifeſt againſt thoſe, who ſlight thoſe things whereof they are ignorant, of how great value and dignity this knowledge is, which the wiſeſt men of all ages have, with incredible ſtudy, labour'd to attain unto, and become poſſeſs'd of. Moreover, I muſt own that *Peter Ramus*'s labours have been of great ſervice to me in the com-

compiling of this account, who in the whole firſt book of his inſtitution, which is not a little one, hath out of *Proclus*, *Laertius*, *Gellius*, *Polybius*, *Tzetzes*, and others, compoſed a mathematical hiſtory both accurately and copiouſly.

The mathematical ſciences were the firſt of all other amongſt men, if we may believe *Joſephus*. He Book I. Chap. 3. writeth, That the poſterity of *Seth* obſerved the order of the heavens, and the courſes of the ſtars. And leſt theſe inventions ſhould ſlip out of the knowledge of men, *Adam* having predicted a two-fold deſtruction of the earth, one by a deluge, the other by fire, they raiſed two columns, one of bricks, of ſtone the other; and inſcribed their inventions upon them, that if the brick one ſhould happen to be deſtroyed by the deluge, that of ſtone, which would remain, might afford men an opportunity of being inſtructed, and preſent to their view the things which it had inſcribed on it. They ſay alſo, that that ſtone pillar, which even in our days is ſeen in *Syria*, was dedicated by them. Thus *Joſephus* ſays; whom I leave to vouch for the ſtory.

That the *Aſſyrians* and *Chaldeans* were the firſt of mortals, after the flood, who applied themſelves to the *mathematics*, is delivered by the ſame *Joſephus*; as alſo by *Pliny*, *Diodorus* and *Cicero*. But the *mathematic arts* which firſt ſprang among the *Chaldeans*, among whom they flouriſhed, were afterwards transferred out of *Chaldea* and *Aſſyria* unto the *Egyptians*, by *Abraham*. For, ' when, at the command of

GOD,

GOD, he went forth from his native ſoil into *Paleſtine*, and from thence into *Egypt*, and perceiv'd the *Egyptians* to be taken with the ſtudy of good arts, and to be of a very notable wit and capacity for learning, (as *Joſephus* teſtifies, *Book* 1. Chap. 9.) he communicated to them arithmetic and aſtronomy; and conſequently geometry, which muſt of neceſſity go before aſtronomy. In which ſtudies afterwards the *Egyptians* ſo flouriſhed, that *Ariſtotle*, 1 *Metaph.* Chap. 1. doth affirm, that the *mathematic arts* were firſt found out in *Egypt*, by their prieſts; who, by their employments, were at leiſure for theſe things.

Then theſe arts croſſing the ſea out of *Egypt*, came to the philoſophers of *Greece*: For *Thales* the *Mileſian*, who flouriſhed 584 years before Chriſt, was the firſt of the *Greeks* who coming into *Egypt*, transferred geometry from thence into *Greece*. He it was indeed, who, beſides other things, found out the 5th, 15th and 26th propoſitions of the firſt book. To the ſame are alſo owing the 2d, 3d, 4th, 5th, of the fourth book. The ſame perſon began to obſerve the equinoxes and ſolſtices, as *Laertius* teſtifies; and he was the firſt who foretold an eclipſe of the ſun, as *Hippias* and *Ariſtotle* do write; and *Tzetzes* ſaith, That he alſo foreſhew'd an eclipſe of the moon to king *Cyrus*. For which thing's ſake he is to be looked on as the firſt founder and author of the mathematical ſciences in *Greece*.

After him was *Pythagoras* of *Samos*: Which moſt antient philoſopher, exceedingly improved

and

and adorned the mathema[illegible]k sciences. And he so gave himself to arithmetick in particular, that almost his whole method of philosophizing was taken from numbers. And he first of all, as *Laertius* relates, abstracted geometry from matter; in which elevation of the mind, he found out the 32d, 44th, 47th and 48th propositions of the first book. But he is especially celebrated for the invention of *Prop.* 32, and 47. of that book; and he conceived so great joy upon this invention, that, as *Apollodorus* witnesses in *Laertius*, on that account he sacrificed an hecatomb. The same person first laid open the matter of incommensurable magnitudes, and the five regular bodies. The same person did both most diligently teach and exercise the art of astrology and music: For he did not only acutely and subtilly find out many things himself, but he also first opened a school, in which youth might learn these honourable and noble arts.

Pythagoras was followed by *Anaxagoras* of *Clazomenæ* and *Oenopides* of *Chios*, of whom *Plato* makes mention in his dialogue, *The Lovers*, where young men are brought in contending about *Anaxagoras* and *Oenopides* in their descriptions of circles. *Aristotle* reports, that a certain treatise of geometry was written by *Anaxagoras*; and we have it from *Laertius*, that it was shewed by him that the sun is greater than *Peloponnesus* (a notable instance of the infancy of astronomy at that time) and that he made some conjectures concerning habitations in the moon. As for *Oenopides*, to him *Proclus*

clus aſcribes the 12th and 13th, *L. 1.* Theſe were followed by *Bryſo*, *Antipho*, and *Hippocrates* of *Chios*; all of them for attempting the quadrature of the circle reprehended by *Ariſtotle*, and at the ſame time celebrated. But amongſt them, *Hippocrates* was by far the moſt famous; that celebrated perſon, who, of a merchant growing to be a philoſopher and a geometrician, beſides the quadrature of the circle, alſo firſt attempted the doubling of the cube, by two mean proportionals, which, as being an excellent, and indeed the only way, all that have followed him to this time have embraced it. 'Tis alſo his peculiar and great commendation, that he, as *Proclus* teſtifies, firſt wrote elements, and digeſted into order the diſcoveries made by others.

Democritus was admirable, not in philoſophy only, but alſo in the mathematicks. His phyſical monuments, and, if ſuch there were, his mathematical works alſo, are wholly loſt, through the envy (as ſome report) of *Ariſtotle*, who deſired to have no other writings read, but his own. The philoſophy of *Democritus* hath been reſtored by *Peter Gaſſendus*, in a moſt learned work lately put forth. *Theodorus Cyrenæus*, although none of his mathematical inventions are extant, yet is great upon this account, if there were no other, that he is reported to have been the maſter of *Plato*.

Unto *Plato* therefore we are come at length, than whom no one brought greater luſtre to the mathematical ſciences. He amplified geometry with great and notable additions, beſtowing

ſtowing incredible ſtudy upon it. And above all, the art analytic, or of reſolution, was found out by him, the moſt certain way of invention and reaſoning. He ſet off and illuſtrated his books of philoſophy in a mathematical way, and encouraged whatſoever was admirable in mathematical philoſophy. Upon the door of his academy was read this inſcription: *υδεις αγεωμετρητος εισιτω*, *Let no one ignorant of geometry enter here*; an illuſtrious inſtance to demonſtrate, how the mathematicks are not foreign, but proper, not unuſeful, nor unbecoming, but honourable and profitable to ſound and certain philoſophy. In a word, how great both admirer and maſter of the mathematicks *Plato* was, that man will of himſelf eaſily underſtand, who ſhall read his monuments through.

Out of *Plato*'s academy, almoſt innumerable mathematicians came forth. Thirteen of *Plato*'s familiar acquaintance are commemorated by *Proclus*, as men by whoſe ſtudies the *mathematicks* were improv'd. From hence were *Leodamus* the *Thaſian*, *Architas* the *Tarentine*, *Theætatus* the *Athenian*, by whom the *mathematics* were notably enlarged. *Leodamus* practiſed the analyſis received from *Plato*, and is ſaid by *Laertius* to have found out many things by the help of it. As for *Theætetus*, both his own inventions, among which are the elements written by him, and the inſcription of regular bodies; and *Plato*'s encomiums, who alſo inſcribed a dialogue to his name, do make him famous.

Archytas

Archytas alſo wrote elements himſelf; and his doubling of the cube is mentioned by *Eutocius*; whoſe ſingular commendation it likewiſe was, that he was almoſt the firſt that brought down the *mathematics* to human uſes; by whoſe contrivance alſo a wooden pigeon was made to fly, as *Gellius* reports; he being preceded by *Dædalus*, and followed by other artificers, yielded matter for the fables of the poets. Moreover, *Archytas* was both a mathematician and general of an army. He five times commanded the forces of his own citizens, in the wars of his country, and five times overcame their enemies. The meer name of *Neoclides* is only famous, he being more illuſtrious for his ſcholar *Leon* perhaps, than for his own inventions. *Leon* certainly wrote elements of all the *mathematics*, improved them, and made them more fit for uſe. Wherefore he is deſervedly to be reckoned amongſt the chief compilers of elements.

Eudoxus of *Cnidos* was not inferior to *Leon*: A man great in arithmetic, and to him, (if we may believe the *Greek* ſcholiaſt) we owe the whole fifth book. He likewiſe wrote elements, and made them more general and encreaſed the ſections begun by *Plato*; over and above this he was the firſt framer of aſtronomical hypotheſes, and derived down the ſprings of geometry, as *Archytas* had done before, to mechanicks. *Amyclas* the *Heracleot*, and *Menæchmus*, and his brother *Dinoſtratus*, *Helicon* of *Cyzium*, *Theudius*, *Hermotimus* the *Colophonian*, *Philippus* the *Medmæan*, all Platoniſts rendered geometry

metry much more perfect. *Menæchmus* also found out the *Conic* sections, and by the help of them, two mean proportionals; whose invention in this case is preferred by *Eutocius* before any other. *Theudius* and *Hermotimus* made the elements more universal and full. And all these, who were of *Plato*'s academy, brought mathematic philosophy to perfection, as *Proclus* saith. *Xenocrates* also, one of *Plato*'s auditors and master of *Aristotle*, as well as *Aristotle* himself, were famous for the knowledge of the mathematicks. When a certain person, who knew nothing of geometry, was minded to be his auditor, *Go thy way*, saith he, *for thou wantest the very handles of philosophy*.

But of *Aristotle*, what can I say; All his books are fill'd with mathematical arguments, out of a collection of which *Blancane* hath made a book. Two of *Aristotle*'s school are especially celebrated, *Eudemus* and *Theophrastus*. This latter wrote two books of numbers, four of geometry, and one of indivisible lines: The other, composed a mathematical history; and from him *Proclus*, and others have borrowed theirs. To *Aristeus*, *Isidore*, *Hypsicles*, most subtil geometricians, we are especially indebted for the books of solids. Lastly, *Euclid* gathered together the inventions of others, disposed them into order, improved them, and demonstrated them more accurately, and left to us those *Elements*, by which youth is every where instructed in the mathematicks. He died in the year before Christ, 284. There followed *Euclid* almost an 100 years afterwards *Eratosthenes* and *Ar-*

chimedes

chimedes. The name of *Eratoſthenes* was very famous, but his writings are loſt. Many remains we have of *Archimedes*, and many we have loſt.

But when I name *Archimedes*, I conceive in my mind the very top of human ſubtilty, and the perfection of the whole mathematical ſciences. His wonderful inventions have been delivered to us by *Polybius*, *Plutarch*, *Tzetzes* and others. *Conon* was contemporary to *Archimedes*, one who was both a geometrician and an aſtronomer, whoſe death *Archimedes* laments in his book of the quadrature of the *Parabola*. There followed *Archimedes* and *Conon*, and that at no great diſtance, *Apollonius* of *Perga*, another prince in geometry, who was called by way of high encomium, *The great geometrician*. There are extant four [now ſeven] moſt ſubtil books of his *Conics*. To the ſame perſon are aſcribed the fourteenth and fifteenth books of *Euclid*, which were contracted by *Hypſicles*. *Hipparchus* and *Menelaus* wrote, this latter, ſix, the other, twelve books of ſubtenses in a circle; for which invention ſo very profitable and neceſſary, great commendations and thanks are due to both. There are alſo extant three books of *Menelaus* concerning ſpherical triangles. Three moſt uſeful books of ſpherics of *Theodoſius* the *Tripolite* are alſo in the hands of all. And theſe indeed, if you except *Menelaus*, lived all of them before Chriſt.

In the year after Chriſt, 70, there appeared in the world, *Claudius Ptolomæus*; the prince of

of aſtronomers, a man certainly wonderful, and as (*Pliny* ſaith) above the nature of mortals. But he was not only moſt ſkilful in aſtronomy, but in geometry alſo, which, as many other things written by him do witneſs, ſo eſpecially do the books of ſubtenſes: Thoſe of *Menelaus*, which were ſix, and the twelve of *Hipparchus*, all contracted by him into five theorems. As for *Plutarch*, a moſt famed philoſopher, there are extant his mathematical problems. And all know of the learned commentaries of *Eutocius* the *Aſcalonite* upon *Archimedes*. By him are recited the inventions of *Philo*, *Diocles*, *Nicomedes*, *Sphorus*, *Heron*, as of ſo many excellent maſters in the mathematicks, concerning doubling the *Cube*. *Heron*'s genius certainly was excellent, as well for mechanics as geometry. The doubling the *Cube* delivered by him, is commended by *Pappus*, Book III. *Prop*. 7. before all other. The admirable works of *Cteſibius* the *Alexandrian*, to whom we owe our pumps, are celebrated by *Vitruvius*, *Proclus*, *Pliny* and *Athenæus*. The name alſo of *Geminus* is not in the loweſt place amongſt mathematicians, whom *Proclus* has preferred in many things before *Euclid* himſelf.

Diophantus, and he alſo an *Alexandrian*, was as great in arithmetic, as *Archimedes*, *Apollonius* or *Euclid* in *geometry*; he was certainly a maſter of all ſubtility relating to numbers: By him was found out that admirable art, which we call *Algebra*; which in theſe times has been rendered more perfect and univerſal by *Francis Vieta*, and *Renatus Carteſius*. There are others

who

who are celebrated amongſt the antients alſo; as *Nicomachus*, famous for arithmetical, geometrical and muſical monuments; *Serenus* well known to geometricians for his two books, concerning the ſection of a cylinder; *Proclus*, *Pappus*, *Theon*. How great a mathematician *Proclus*, was, is manifeſt from his learned commentaries on *Euclid*, and other writings. And this is he, as I ſuppoſe, who, as *Zonarus* reports, and from him *Ramus* and *Baronius*, about the year of Chriſt 514, with optic artifice, and the glaſſes which he uſed, burnt the fleet of *Vitalian*, who was beſieging *Conſtantinople*. The praiſes of *Theon*, which truly are deſervedly great, *Peter Ramus* wonderfully exaggerates; inſomuch, that even the books which hitherto all have aſcribed to *Euclid*, ought, as he thinks, to be attributed to *Theon*. But *Ramus*, who every where is ready to detract from *Euclid*, and this without grounding himſelf upon any ſolid foundation, is not to be hearkened to here. To come at length to a concluſion, let *Pappus* bring up the rear, the laſt in time amongſt the antients, as being one who lived about the year 400, but in reputation, and all mathematical commendation, to be reckoned among the firſt. *Alexandria*, that city ſo fruitful of great men, which before had brought forth *Hypſicles*, *Cteſibius* and *Diophantus*, produced him alſo, to the great advantage of the mathematicks. He wrote ſeven books of mathematical collections, of which the two firſt are loſt. The five other do abound with ſo many, and ſuch various moſt noble inventions

in

in almoſt all parts of the mathematicks, that they are eſteemed amongſt the chief monuments of the antients which are extant.

And thus you have a ſhort hiſtory of the origin and progreſs of the mathematicks, from which appears the antiquity, excellence, and dignity of this ſcience. Certainly the ſame eminent perſons in the commonwealth of learning, who diſcovered philoſophy, diſcovered alſo the mathematicks, like two ſiſters born at one birth; whom, if any one would violently ſeparate from each other, he certainly attempts to break off their native concord, with moſt notable injury, and as it were cruelty to both; ſeeing, as it is wont to fall out in the caſe of twins, where they are removed from one another, in place or by death, ſo it will be like to happen here, that mathematicks being plucked away from her, philoſophy muſt needs languiſh and pine away.

N. B. Q. *E. D.* or *W. W. D.* is which was to be demonſtrated.

Q. *E. F.* which was to be done.

Q. *E. I.* which was to be found.

DOCTOR BARROW's WORDS

PREFIXED BEFORE HIS

APOLLONIUS.

GOD ALWAYS ACTS GEOMETRICALLY.

HOW great a geometrician art thou, O Lord! for, while this ſcience has no bound; while there is for ever room for the diſcovery of new theorems, even by human faculties, thou art acquainted with them all at one view, without any train of conſequences, without any wearisome application of demonſtration: In other arts and ſciences our underſtanding is able to do almoſt nothing; and like the imagination of brutes, ſeems only to dream of ſome uncertain propoſitions; whence it is, that in ſo many men are almoſt ſo many minds: but in theſe geometrical theorems all men are agreed: in theſe the human faculties appear to have ſome real abilities, and thoſe great, wonderful, and amazing. For thoſe faculties which ſeem of almoſt no force in other matters, in this ſcience appear to be efficacious, powerful and ſucceſsful, &c. Thee therefore do I take hence occaſion to love, and rejoice in, and admire; and to pant after that day, with the earneſt breathings of my ſoul, when thou ſhalt be pleaſed, out of

thy

thy bounty, out of thy immense and sacred benignity, to grant me the favour to perceive, and that with a pure mind, and clear vision, not only these truths, but those also which are more numerous, and more important; and all this without that continual and painful application of the imagination, which we discover these withal, &c.

Mathematical Notes, or Abbreviations.

= The note for equality. So $a = b$ signifies that a and b are equal.

+ The note for addition. So $a + b$ signifies the sum of a and b together.

— The note for subtraction. So $a - b$ signifies the difference between a and b.

× The note for multiplication. So $a \times b$, or ab signifies a multiplied by b.

:: The note for equality of proportion. So $A : B :: a : b$ signifies that A bears the same proportion to B, that a bears to b.

∸ The note of continued proportion. So ABC ∸ signifies that A bears the same proportion to B, that B bears to C.

q The note for a square. So CBq signifies the square of the line CB.

c The note for a cube. So CBc signifies the cube of of the line CB.

THE

THE

ELEMENTS

OF

EUCLID.

BOOK I.

DEFINITIONS.

1. A Point is a mark in magnitude, which is (ſuppoſed to be) indiviſible.

That is, which cannot be divided ſo much as in thought. A point is the beginning, as it were, of all magnitude, as unity is of number.

2. A line is a magnitude which hath length only, and wants all breadth; foraſmuch as it is underſtood to be produced from the flowing of a point. *Fig.* 1. *Table* 1.

3. Points are the terms of a line.

4. A right line is that which lies evenly betwixt its terms.

Or, as *Archimedes*; a right line is the leaſt of all thoſe which have the ſame terms; or, is the ſhorteſt of all thoſe which can be drawn betwixt two points.

Or, as *Plato* hath it; a right line is that whoſe extremes hide all the reſt: (that is, when the eye is placed in a continuation of the line.)

The ſenſe is the ſame in all. The inſtrument whereby right lines are deſcribed, is (called) a rule; which, whether it be ſtraight or not, you may know by this trial.

Deſcribe a line, according to the rule; then, turning the rule ſo that that which was before the right-hand end may now become the left-hand end, apply it again to the

the line before deſcribed; if it doth now entirely fall in with the line, the rule is ſtraight; if not, the rule is not ſtraight. The reaſon hereof depends on *Axiom* 13.

5. A ſurface is a magnitude which hath only length and breadth.

It hath two dimenſions therefore: and is underſtood to be produced by the flowing of a line.

6. Lines are the extremes of a ſurface.

7. A plane, or a plain ſurface, is that which lies evenly betwixt its extreme lines.

Or, as *Hero*, that to all the parts whereof a right line may be accommodated.

For it is produced from the motion of a right line.

Or, a plain ſurface is that whoſe extremes any of them hide all the reſt, (the eye being placed in a continuation of the ſurface.)

Or, it is the leaſt of all ſurfaces, which have the ſame terms. The ſenſe is the ſame in all.

Euclid hath not here defined a body or ſolid, becauſe he was not yet about to treat concerning it. But leſt any one ſhould want the definition thereof, take it here thus: A body is a magnitude, long, broad and deep. A body therefore, hath three dimenſions; a ſurface two; a line one; a point none.

8. A plain angle is the mutual inclination to each other of two lines, which touch one another in a plane, and ſo as not to make one line.

Fig. 2, 4. Therefore the two lines A B, C A touching one another in A, but ſo as not to make one line, conſtitute an angle.

9. The ſides, or legs of an angle are the lines which make the angle.

10. The *vertex* or top of an angle is the point (A) in which the legs do meet, and touch one another.

Note, That a ſingle angle is deſigned by one letter put at the top: when there are more at one point, they are deſign'd by three letters, the middlemoſt of which denotes the top of the angle; and many times alſo by one letter interpos'd betwixt the ſides near the top. So in *Fig*. 5. the angle made by the lines B A, C A is deſigned either by three letters B A C, or by one only O.

11. Angles are called equal, if when the tops of them are laid upon one another, the ſides of one agree with the ſides of the other. But unto this it is not required that the ſides ſhould be of an equal length.

12. They

12. They are called unequal, when, the top and one side agreeing, the other doth not agree and that is called the greater, whose side falls without. So the angle B A E is greater than the angle B A C. *Fig.* 5.

An angle is not diminish'd or increas'd by the diminution or augmentation of the sides that include it.

13. A right lin'd angle is that which right lines constitute; a curvi-lineal, which crooked lines; a mixt one, that which a right line and a crooked one make. *Fig.* 2. 4.

14. When the right line (C A) standing upon the right one (B F) leans unto neither part, and therefore makes the angles on both sides equal, C A B = C A F both of the equal angles are called right ones: But the right line C A which stands upon the other, is called a perpendicular line, or barely a perpendicular. *Fig.* 6.

A right angle may also be defined thus. *Fig.* 6.

A right angle is that, that (B A C) when on the other side an equal one ariseth (C A F) if you produce or draw forth a side, as (B A.)

Two rules so joined as to contain a right angle, make an instrument which is called a square. *Pythagoras* was the inventor of it, as *Vitruvius* affirmeth *c.* 2. *l.* 9. So great is the use and force of a right angle in framing, measuring and strengthening all things, that nothing almost can be done without it. The proof of a square is made thus: Apply the side of it, A E, to the right line A F and describe the right line C A along the other side: Then turning the square towards B, if on both sides it agrees to the right line C A, A B, you may know that it is true and exact. The reason hereof appears from the 14th *Definition* itself.

15. The angle B A C, which is greater than the right one F A C, is called an obtuse angle. *Fig.* 7.

16. The angle (L A C) which is less than the right angle (F A C) is called an acute one. *Fig.* 8.

17. A plain figure is a plain surface, bounded on every side with one or more lines.

18. A circle is a plain surface, contained within the compass of one line, called the circumference; from which line all the right lines that can be drawn unto one certain point, within the contained space (A) are equal. *Fig* 9.

19. That point is called the center.

20. The diameter is a right line (B A) drawn thro' the *Fig.* 9.

the center, and on both ſides ended at the circumference; and conſequently it divides the circle into two equal parts. (As is abundantly manifeſt from the exact agreement of two ſemi-circles when laid upon one another.)

21. The ſemi-diameter, or *radius*, is the right line A F, drawn from the center to the circumference.

22. The ſemi-circle is a figure (B L C) which is contained by the diameter B C, and half the circumference (B L C).

Mathematicians are wont to divide the circumference into 360 equal parts (which they call degrees) the ſemi-circumference into 180, the quadrant, or quarter, into 90.

23. A right-lin'd figure, is a plain ſurface bounded on every ſide with right lines.

Fig. 10. 24. A triangle is a plain ſurface contained by three right lines.

This is the firſt and moſt ſimple of all right-lin'd figures and that into which they are all reſolved.

Fig. 10. 25. An equilateral triangle, is that which hath all the ſides equal.

Fig. 11, 12. 26. An *iſoſceles*, or equicrural triangle, is that which hath only two ſides equal.

Fig. 13. 27. A *ſcalenum*, is that which hath three unequal ſides.

Fig. 13. 28. A right-angled triangle, is that which hath one angle right.

Fig. 12. 29. An obtuſe-angled triangle, is that which hath one obtuſe angle.

Fig. 10, 11. 30. An acute angled triangle, is that which hath three acute angles.

Fig. 14, 15. 31. Amongſt quadrilateral figures, the rectangle is that which hath four right, and conſequently equal angles; whether the ſides be equal or not.

Fig. 15. 32. A ſquare, is that which hath equal ſides, and is right angled, and conſequently equi-angled.

Every ſquare is a rectangle; but every rectangle is not a ſquare.

Fig. 16. 33. A rhombus is a quadrilateral, or four-ſided figure, which is equilateral, but not equi-angled.

Fig. 17. 34. A rhomboides, is that which hath the oppoſite ſides and angles equal, but is neither equilateral, nor equi-angled.

Fig. 14, 15, 16, 17. 35 A parallelogram, is a quadrilateral figure, which hath each two of its oppoſite ſides (A B, F C and B F, A C) parallel

parallel to each other. Now what parallel lines are, will be ſhewed in the following *definition*.

Every rectangle and ſquare is a parallelogram; but every parallelogram is not a rectangle or a ſquare.

36. Right lines are parallel, or equi-diſtant, which being in the ſame plane, and drawn out on both ſides infinitely, are diſtant from one another by equal intervals. *Fig.* 18.

The intervals are ſaid to be equal; in reſpect of the perpendiculars. Wherefore if all the perpendiculars (Q L) unto one of the two parallels (A B) ſhall be equal, the right lines (A B, C F) are ſaid to be parallel.

Parallels are produced, if the right line (L Q) which is perpendicular to the right line (A B) be moved along (A B) always perpendicularly; for then its extremities L deſcribes the parallel C F.

37. The diameter, or diagonal of a parallelogram, and every quadrilateral, is a right line (A F) drawn through the oppoſite angles. *Fig.* 17.

38. Plain figures contained by more ſides than four, are called many-ſided, or many-angled, and by a *Greek* word, *polygons*.

39. The external angle of a right-lin'd figure, is that which ariſes without the figure, when the ſide is produced. Such are F B C, G C A, H A B. Every figure therefore hath ſo many external angles as it hath ſides, and internal angles. *Fig.* 19.

Poſtulates.

A Poſtulate is that which is manifeſt in itſelf, that it may eaſily be done, or conceived to be done. It is required therefore to be granted that we may,

1. From any point given draw a right line unto any other point given.
2. Draw forth a finite right line in length ſtill farther.
3. From any center at any interval deſcribe a circle.

Axioms.

AN *Axiom* is a truth manifeſt of itſelf.

1. Thoſe things which are equal to the ſame thing, are equal alſo amongſt themſelves. And that which is greater or leſs than one of the equals, is alſo greater or leſs than the other of them.

2. If to equals you add equals, the whole will be equal.

3. If from equals you take away equals, the remainders will be equal.

4. If to unequals you add equals, the wholes will be unequal.

5. If from unequals you take away equals, the remainders will be unequal.

6. What things are each of them half of the ſame quantity, are equal amongſt themſelves; and what things are double or treble, or quadruple of the ſame, are equal amongſt themſelves.

7. What things do mutually agree with one another are equal.

8. If right lines be equal, they will mutually agree with one another; and the ſame thing is true of angles.

9. The whole is greater than its part.

10. All right angles are equal amongſt themſelves.

11. Parallel lines have a common perpendicular: that is, the right line which is perpendicular to one of them, is perpendicular alſo to the other.

Fig. 21. 12. The two perpendicular lines (L O, Q I) intercept equal parts of the parallels, L I, O Q.

13. Two right lines do not comprehend a ſpace; for [illegible] there are required three at the leaſt.

14. Two right lines cannot have one common ſegment; for that they cut one another only in a point.

Of propoſitions ſome propoſe ſomething to be done, and are called *problems*; in others we proceed no further than bare contemplation, which therefore are named *theorems*.

PRO-

PROPOSITIONS.

THE requisite citations are found in the margin. When propositions are cited, the first number designs the proposition; the letter *l*, with the number following, signifies the book. As when you meet with *(per 5. l. 3.)* you must read it thus, (By the fifth proposition of the third book.) The figure is always to be sought amongst the figures of that book in which we are then conservant. The rest of the citations are easy to be understood.

The primary affections of triangles and parallelograms are delivered in this book. The more famous propositions are, 32, 35, 37, 41, 44, 45, 47.

PROPOSITION I. Problem.

UPON *a given right line (AB) to make an equilateral triangle.* *Fig.* 23.

From the center A, with the interval (A B) *(a)* describe the circle F C B; and from the center B with the same interval B A describe the circle A C E, cutting the former in the point C, from which point draw the right lines C A, C B. (a) *Per Postul.* 3.

I say, that the triangle A C B now made, is equilateral. For the right line A C is equal to the right *(b)* line A B, seeing they are semi-diameters of the same circle F C B. And again, the right line B C is equal to the same right line B A, seeing they are both semi-diameters of the circle L C A. Therefore, A C, B C are *(c)* equal betwixt themselves. And therefore all the sides of the triangle are equal. Therefore the triangle *(d)* A C B is both equilateral, and made upon the given line A B; which was the thing to be done. *Q. E. F.* (b) *Per Def.* 18. (c) *Per Axiom* 1. (d) *Per Def.* 25.

Corollary. "Hence we may measure an inaccessible "line, as A B. For suppose an equilateral triangle "whatsoever B D E applied to the point B, along the line "B A. Looking from the point B, along the line B E, "mark as many points as you conveniently can in the line "B C. Then remove the triangle B D E along the line "B C, from one place to another of that line, until, by *Fig.* 77.

"taking

" taking aim along the side of the triangle ED or CF, " you see the inaccessible point A in a continuation of that " line. Thus the triangle BAC is as well equilateral " as BDE. If therefore you shall now measure the ac- " cessible line BC, you have the measure of the inaccessi- " ble AB. *Q. E. F.*

PROP. II. Problem.

Fig. 24. *FROM a given point A, to draw a right line equal to one given, EF.*

Take with a pair of compasses the interval EF, and transfer it from A to D, the right line AD will be equal to the given EF.

PROP. III. Problem.

Fig. 24. *TWO unequal right lines being given, from the greater of them GH to cut off GI equal to the less EF.*

Take with a pair of compasses the interval of the less given line EF, and transfer it unto the greater from G to I.

PROP. IV. Theorem.

Fig. 25. *IF in two triangles (X, Z) one side of the one (BA) be equal to one side (FL) of the other, and another side (CA) of the one equal to another side (IL) of the other, and the angles (A and L) made by those sides be also equal; then the bases (BC, FI) are likewise equal as also the angles at the bases (BF and CI) which are opposite to equal sides, and consequently the whole triangles are equal.*

For if we ſuppoſe the triangle Z to be laid upon the triangle X, the ſides L F, L I will perfectly agree and fall in together with the ſides that are equal to them, A B, A C, and this in ſuch ſort (*c*) that the three points (L F. I.) ſhall fall upon three points (A. B. C.) Therefore the whole baſe F I will alſo fall upon the whole baſe BC. But then the angles F, B, and likewiſe thoſe, I, C, and the whole triangles will mutually *(congruere)* agree to each other. All therefore, by the 7th *Axiom*, are equal. *Q. E. D.* Which was the thing to be demonſtrated. (c)*Per Axiom* 8.

Fig. 8.

Corollary. (1.) " Hence we may alſo in another way " meaſure the line A B, although otherwiſe impracticable " by reaſon of ſome obſtacle, as a river, &c. between " the extremities thereof. For from any point whatſo-" ever, as the point C, let the angle A C B be obſerved, " and then let the lines A C, B C be meaſured; and in " any acceſſible plane let there be meaſured about the " angle F, which is equal to the angle C, two lines F D " and F E, which are equal to the lines A C and B C " reſpectively. And there will be the acceſſible line D E " equal to the inacceſſible A B. *Q. E. I.*

Fig. 79.

Corollary (2.) " Hence alſo thoſe who play at bil-" liards with ivory balls, may learn how by the reflection " of their own to hit and remove their adverſary's ball. " For let B be the ball to be ſtricken, A that which is to " ſtrike it, and C D the rectilinear plane. Let the line " B E be perpendicular to the line C D, and D E be " equal to D B. If the ball A be ſtricken, and carried a-" long the right ſide A F E unto the point F, it will there " be ſo reflected, that after the reflexion it will tend unto " B. For in the triangles B F D, E F D, the ſide F D " is common to both, and the ſide B D is equal to the " ſide D E; and the angles at D are equal, as being " right ones. The whole triangles therefore are equal; " and therefore the angle B F D, which is equal to the * " angle D F E, is * equal to A F C, the angle A F C " being vertically oppoſite to D F E. Wherefore, ſeeing " the angle A F C is the angle of incidence, which in " ſuch caſes is equal to the angle of reflection, it is mani-" feſt that B F D, which hath been proved equal to A F C, " is the angle of the reflection of the ball A, and that " the

* *Per* 15. *l.* 1.

"the ball tending towards E, is in the point F so reflected, "as to hit the ball B. *Q. E. D.*

Scholium or observation.

BY much the same way of reasoning, whereby this 4th *Proposition* has been demonstrated, the following *Theorem*, which we shall have occasion to use by and by, may be demonstrated also.

Fig. 25. If in two triangles X, Z, the sides BC and FI shall be equal, and the angles adjacent to these two sides equal also, *viz.* B and C equal to F I; all the other things, and the whole triangles themselves will be equal.

For the side FI laid upon the side BC will agree, or throughly coincide with it (*a.*) And then because the angles B and C are equal to those F and I, when the side FI is laid upon the side BC, FL (*b*) will fall exactly upon BA, and IL upon CA. Therefore the point L will fall upon the point A (for if it fall without A, the sides FL, IL would not fall upon the sides BA, CA.) Therefore all things are equal, by the 7th *Axiom.*

(a) *Per Axiom* 8.
(b) *Per Axiom* 8.

PROP. V. Theorem.

Fig. 26. *IN an isosceles, or equicrural triangle, the angles at the base* (*A, C*) *are equal.*

Let the triangle ABC be understood to be twice put, but in an inverted posture *c b a.* Because therefore, in the two triangles ABC, *c b a*, the side AB is by the supposition, equal to the side *c b*, and the side CB to the side *a b*, and the angle B to to the angle *b*; the angle A also at the base (*c*) will be equal to the angle *c. Q. E. D.* For as for the angles C and *c* they are the same.

(c) *Per* 4. *l.* 1.

Corollary.

THEREFORE an equilateral triangle is also equiangular.

PROP. VI. Theorem.

IF in a triangle (ABC) two angles (A and C) be equal, the sides also (AB, BC) which are opposite to those angles, are equal also. Fig. 6.

Let the triangles A B C be supposed to be twice put but in an inverse situation, *c b a*. Because therefore in the triangles A B C, *c b a*, one side A C is equal to one side (*c a*) and the angle A is equal to the angle *c*, and the angle C equal to the angle *a*, all the other things shall be likewise (*a*) equal, and consequently A B shall be equal to the side *c b*. Q. E. D. For as for the line C B and *c b* they are the same.

(a) *Per Schol. Prop.* 4.

Corollary.

THEREFORE an equiangled triangle, is also equilateral.

Corollary (2.) " Hence, by the means of the shadow of " the sun, we may measure the height of a tower, or " any elevated point. For when the sun is elevated 45 " degrees above the horizon, the shadow which the " tower cast towards the horizon, will be exactly equal " to its height. For, by reason that the angle A C B is " half a right angle, the angle B A C also * will be half " a right one; and so by the force of the present propositi- " on, the line A B will be exactly equal to the line B C. " The line B C therefore being found by measuring, there " is found at the same time the line A B, the height of the " the tower above the horizon.

Fig. 80.

* *Per Corol.* 11. *Prop.* 32. *l.* 1.

Corollary (3.) " The same thing also may be found " without the sun by the means of an astronomical qua- " drant. For where the angle of elevation is half-right, " there the height of the tower above the observer's eye is " equal to the distance of the same eye, from that part " of the tower which is opposite to it. The distance there- " fore of the eye from the tower being given by measuring, " there is given at the same time the height of the tower. " *Q. E. I.*

The

The ſeventh propoſition in *Euclid* is for the ſake of the eighth, which without it will here be demonſtrated.

PROP. VIII. Theorem.

Fig. 27. *IF two triangles (X, Z) have all their ſides equal amongſt themſelves reſpectively (AC equal to EF; CB to FI; AB to EI;) they will alſo have all the angles which are oppoſite to equal ſides, equal (C equal to F; A to E; B to I.)*

For ſuppoſe the ſide AB laid upon its equal EI, if then the point C falls upon F, the triangles will in the whole agree or coincide, and conſequently all the angles will be equal. But the point C will fall upon the point F. For,

Fig. 81. "From the center A let a circle be deſcribed with the "ſemi diameter EF; and from the center I, let another "circle be deſcribed with the ſemi-diameter IF, the "point C by the reaſon of the equality of the ſides of both "triangles, will be in the circumference of both circles, "and conſequently in the point E, the common interſec-"tion of both theſe circumferences. *Q. E. D.*

PROP. IX. Problem.

Fig. 29. *TO biſect or divide into two equal parts, a given right lin'd angle, as IAL.*

From the ſides of the angle take with a pair of compaſſes two equal lines, AB, AC; then from the centers B and C deſcribe two equal circles cutting one another in F; which done, draw the line FA. This biſects the angle.

For draw the line BF, CF; the triangles FAB, FAC are to each other equilateral; for the ſides AB, AC are by the conſtruction equal, as in like manner are the ſides BF, CF, they being ſemidiameters of equal circles; and AF is common to both triangles. Therefore

(d) Per 8 l. 1. the angles BAF, CAF (*d*) are equal. Therefore the given angle IAL is biſected. *Q. E. F.*

Corollary.

Corollary.

HENCE we may learn how an angle may be divided into all equal angles, 4, 8, 16, &c. *viz.* by bisecting each part again.

Scholium.

NO one hath hitherto taught the way of dividing angles into all equal parts whatsoever with a pair of compasses and a rule.

Yet may you divide any given angle mechanically into any equal parts whatsoever, if from the top of the angle, as the center, you describe an arch between the legs of the angle, and divide the arch into as many equal parts as you require; for right lines let down from A, through the points of the division, will cut the angle into so many equal parts. *Fig.* 30.

PROP. X. Problem.

TO *bisect a finite given line* (*AB.*) *Fig.* 31.

Upon the given line A B make an equilateral (*a*) triangle A G B, bisect its angle G (*b*) with the right line G C. The same shall bisect the given line A B. (a) *Per* 1. *l.* 1. (b) *Per præced.*

For in the triangles X, Z, the side C G is common; and by the construction G B, G A are equal and the angles contained between them A G C, B G C, are likewise equal. Therefore the bases A C, B C (*c*) are equal. The given line therefore A B is bisected. Q. *E. F.* (c) *Per* 4. *l.* 1.

But for practice, it is sufficient, from the centers A and B, to describe two equal circles cutting one another in G and L, and so to draw the right line G L.

PROP. XI. Problem.

F*ROM a given point* (*A*) *to raise a perpendicular in a given right line* (*L I.*) *Fig.* 32.

With a pair of compasses take the equal lines A C, A F. From the center C and F describe two circles, cutting

cutting one another in B. The line which is drawn from B to A will be the perpendicular required.

For let the right lines CB, FB be drawn. The triangles X and Z are equilateral to one another. Therefore
(a) *Per* 8. *l.* 1. the angles CAB, FAB are equal (*a.*) Therefore BA is (*b*) perpendicular to the line (LI.) *Q. E. F.*
(b) *Per Def.* 14. In practice this and the next are eaſily performed by the help of a ſquare.

PROP. XII. Problem.

Fig. 33. *FROM a given point (A) which is without an infinite right line (as LQ) to let fall a perpendicular to that line.*

From the center A deſcribe a circle which may cut the
(c) *Per* 10. *l.* 1. given LQ in C and I. Biſect the right line CI (*c*) with the right line AB. This AB is the perpendicular required.

For let there be drawn AC, AI. Becauſe by the conſtruction, the triangles X and Z are equilateral to one
(d) *Per* 8. *l.* 1. another. Therefore the angles (*d*) CBA, IBA, are equal. Therefore AB is (*e*) perpendicular. *Q. E. F.*
(e) *Per Def.* 14.

PROP. XIII. Theorem.

Fig. 34. *THE right line (BA) ſtanding upon the right line (CF) either makes two right angles, or angles equal to two right ones.*

For if BA ſtand upon it perpendicularly, then by the 14th *Definition*, the two angles BAC, BAF will be right ones. And if BA ſtand obliquely, let there be rais'd
(f) *Per* 11. *l.* 1. (*f*) the perpendicular AL. Where, becauſe the unequal angles CAB, FAB poſſeſs the ſame place which the two right ones CAL, LAF do, and agree to them, they
(g) *Per Axiom* 7. are equal (*g*) to them. *Q. E. D.*

Corollaries.

1. IN the ſame manner it will be demonſtrated if more right lines than one ſtand upon the ſame right line, that the angles thereby made are equal to two right ones.

2. Two

2. Two right lines cutting one another, BAC, FAL, make the angles equal to four right ones. *Fig. 37.*

3. All the angles which are about one point, make angles equal to four right ones. It appears from *Corollary* 2. for they are four right ones, cut into more parts. *Fig. 36.*

4. The angle CAF being known, you at the same time know its compliment unto two right angles BAF. For example, Let the angle CAF be of 70 degrees; the angle BAF will be of 110 degrees. For those two numbers added together make 180 degrees, which is the measure of two right angles. *Fig. 37.*

PROP. XIV. Theorem.

IF two right lines (XR, ZR) at the same point of a right line (QR) make the angles on both sides (XRQ, ZRQ,) equal to two right angles; the lines (XR, ZR) make one right line. *Fig. 35.*

If you deny it, Let XR, BR make one right line. Therefore the angles XRQ, QRB (*a*) will make two right angles. Which thing is (*b*) absurd; seeing by the hypothesis XRQ, ZRQ do make two right angles. (a) *Per* 13. *l.* 1. (*b*) *Contra, Axiom* 9.

PROP. XV. Theorem.

IF two right lines (BC, FL) cut one another in A, the angles opposite at the top (A) are equal, viz. LAB to CAF, and BAF to LAC. *Fig. 37.*

For because BA stands upon the right line LF, the angles LAB, FAB are (*c*) equal to two right ones: And because FA stands upon the right line BC, the angles FAC, FAB are also equal (*d*) to two right ones. Therefore the two angles together (*e*) LAB, FAB are equal to those two together CAF, FAB; by taking away FAB, common to both, LAB (*f*) remains equal to CAF. In the same manner BAF, LAC are shewed to be equal. (*c*) *Per* 13. *l.* 1. (d) *By the same Prop.* (e) *Per Axiom* 1. (f) *Per Axiom* 3.

Corollary. "From these two propositions we gather in "catoptrics, that a ray of light, as reflected in an angle "equal

Fig. 82. "equal to the angle of incidence, taketh the shortest way "of all. *e.g.* When the angles BED, AEF are "equal, the lines AE and EB taken together, are "shorter than any lines whatsoever, as AF and FB taken "together. For from the point B, let the perpendicular "line BC be let down; and let BD and DC be equal: "Let the lines also EC and FC be drawn. Now in the "triangle BED and DEC, seeing the side DE is com- "mon to both, and the side BD and DC are equal by "the hypothesis, as is also in the like manner BDE equal "to the angle CDE; the triangles also shall be * equal "in all other things, and BE shall be equal to CE, and "the angle BED to the angle DEC; (where, because "the angle DEC is equal to [BED, that is] AEF, the "lines AE, EC are proved to make one right line.) "And in the same manner the line BF will be proved "equal to FC. Seeing therefore the lines BE and EA "taken together, are equal to the line CA, and the lines "BF, FA taken together, are equal to the lines CF, FA "taken together; it is manifest that CA, which is one "side of the triangle ACF * is less than the two sides "CF, FA taken together. *Q. E. D.*"

* *Per* 4. *l.* 1.

* *Per* 20. *l.* 1.

PROP. XVI, XVII.

THESE two propositions are contained in proposition 32; and are not here made use of till then.

PROP. XVIII. Theorem.

Fig. 38. *IN every triangle the angle (A) which is opposed to the greater side (BO) is the greater; and (B) which is opposite to the lesser side (AO) is the lesser angle.*

(a) *Per* 6. *l.* 1. (A) cannot be equal to (B) for then the opposite sides BO, AO would be equal (*a*); which is contrary to the hypothesis. Neither can A be less than B, for if it were so, there might within the angle B be made an angle ABF by the right line BF; which angle should be equal to A. But then by the 6th of this book BF, AF shall be equal; and

and if you add to both OF, then BF, FO shall be equal to AO. But AO by the hypothesis is less than BO. Therefore BF, FO shall be less than BO, which contradicts the definition of a right line, which is the shortest of all betwixt two points. Therefore the angle A is neither less than B, nor equal to it. Therefore it is greater. *Q. E. D.*

PROP. XIX. Theorem.

IN the triangle AOB the side (BO) which is opposed to the greater angle (A) is the greater; And (AO) which is opposed to the lesser angle B, is the lesser. Fig. 38.

This proposition is the converse of the former. BO is not less than AO, for if it were, the angle (A) by the 18th would be less than B; which is contrary to the hypothesis. Nor can BO be equal to AO, for in this case, by the 5th, the angles A and B would be equal. But this equality of those angles is contrary to the hypothesis. Therefore BO is greater than AO. *Q. E. D.*

Corol. " Hence we gather, that a globe, or ball perfectly polished, cannot rest in an horizontal plane perfectly polished, but where it toucheth the earth. For let the line AB be an horizontal plane, C the earth's centre, CA the semidiameter of the earth, perpendicular to the tangent AB. The globe placed at B, because of its gravity, and the declivity of the plane, will descend towards A. For in the triangle CAB, the perpendicular line CA, which is opposite to the acute angle ABC, is less than the line BC, which is opposed to the right angle BAC; and so there is from B to A a perpetual descent, in which the globe cannot rest. And in the like manner we prove the descent of fluids and their conformation into a spherical surface. Fig. 83.

PROP. XX. Theorem.

IN any triangle, any two sides of it taken together, are greater than the remaining side.

This, with *Archimedes*, is, as were an *Axiom*; forasmuch as it is immediately manifest out of his definition of a right line; which see above amongst the definitions.

PROP. XXI. Theorem.

Fig. 39. *IF from the ends of one side AB, two right lines be drawn, and joined together within the triangle, (as the lines AO, BO) these are less than the sides of the triangle (AC, BC) but they comprehend a greater angle (A O B.)*

(a) *Per* 20. *l.* 1. (b) *By the same.*

For as for the first part of the proposition, draw out AO unto F: AC, CF are (*a*) greater than A F. Therefore the common line F B being added, A C, BC are greater than A F, F B. Again, O F, FB are greater (*b*) than O B. Therefore the common A O being added, A F, B F are greater than A O, B O. Therefore A C, C B are much greater than A O, O B.

The second part of this proposition will be demonstrated in the second corollary of the first part of proposition 32. And in the mean while we shall make no use of it.

PROP. XXII. Problem.

Fig. 40. *TO make a triangle of three given right lines (BO, LB, LO) of which any two must be greater than the third.*

Let B L, one of the given lines be taken, and B one of its extremities being taken for the center, with the interval of the other given line B O, describe an arch.

Then the other extremity L being taken for the center, with the interval of the third given line L O, describe an arch, cutting the former in O; which being done, and the right lines B O, L O being drawn, I say that that is done which was to be done.

The demonstration is manifest from the construction.

PROP.

PROP. XXIII. Problem.

AT a giv n point in a right line (as B) to make an angle equal to a given one (A.)

First of all let CF be drawn at a venture, cutting the sides of the given angle A. Then in the given right line from B, take BL equal to AF. Then from the center B describe a circle with the interval AC; afterwards another from the center L with the interval FC which may cut the former in O. Then from O unto B, and L having drawn right lines, the angle LBO will be equal to the the given one A. *Fig.* 41.

For by the construction, the triangles are equilateral to one another. Therefore by the 8th of this book the angles B and A are equal.

Scholium.

IT seems meet for the sake of beginners to propound some things here which are necessary for practice about angles.

The measure of an angle is the arch of a circle, which is described from A the top of the angle as the center. Therefore look how many degrees the arch BC, which is intercepted between the legs of the angle BAC shall contain, of so many degrees the angle BAC shall be said to be. And so because BF, a quarter of the circumference, contains 90 degrees, and measures the right angle BAF, a right angle shall be said to be of 90 degrees. In like manner, because half the circumference, which is divided into 180 degrees, measures two right angles, and the whole circumference, which is divided into 360 degrees measures four right angles; two right angles shall be said to make 180 degrees, and four 360 degrees. These things being premised, the practice about angles is as follows. *Fig.* 41.

1. At B a given point in a right line to make an angle equal to the given one A. *Fig.* 42.

From A, the top of the given angle as the center, describe betwixt the sides the arch CF. Then from B, the given point as the center, describe with the same interval the arch LZ; from which take off LO equal to CF. Through B and O draw a right line; LBO shall be equal to the given A.

2. To

Fig. 3. 2. To examine the degrees of the given angle O P Q. This is done very eaſily through any ſemi-circle or protractor, which is divided into 180 degrees. For put the center of the ſemi-circle upon P, the top of the angle, and the radius of the ſemicircle P L upon the ſide of the angle P Q; and the arch L O, which is intercepted betwixt the legs of the angle, will ſhew of how many degrees the given angle is.

3. To frame an angle, containing a given number of degrees, as 42.

Draw the right line X Q, in which mark the point P. Fig. 43. Upon P put the center of the ſemi-circle, and its ſemi-diameter P L upon P Q. From L number 42 degrees, that is, until you come to O. A right line drawn from P through O, will give the angle O P L of 42 degrees.

PROP. XXIV, XXV. Theorems.

Fig. 44. *IF two triangles (B A C, B A F,) ſhall have two ſides (B A, AC,) equal to two (B A, AF,) one ſide of one, to one ſide of the other; and if one of the triangles hath the angle (B A F) contained by thoſe ſides greater than the other (B AC,) it ſhall have the baſe B F greater than the Baſe (B C.)*

And again, if it hath the baſe greater, it ſhall have the angle greater.

From the center A, deſcribe a circle which paſſeth through C, it ſhall paſs alſo through F, becauſe A C, A F are ſuppoſed to be equal. Therefore B F ſhall fall betwixt the points A and C. Then join C F. The angle B C F is greater than the angle A C F; that is, by the 5th of this book, than the angle A F C, and conſequently much (a) *Per* 19. *l.* 1. greater than the angle B F C. Therefore in the triangle B C F, (*a*) B F, which is oppoſite to the greater angle B C F, is greater than B C, which is oppoſite to the leſſer angle B F C.

2. As for the ſecond part of the propoſition, this is manifeſt from the firſt part, and propoſition 4.

PROP.

PROP. XXVI. Theorem.

IF two triangles (X and Z) have two angles equal to two, one angle of the one, equal to one angle of the other (B to F and C to I,) and one side of one equal to one of the other, whether it be that which is betwixt the equal angles (as BC=FI) or a side which is opposed to one of the equal angles (as AC=LI,) all the other parts shall be equal. Fig. 25.

For in the place, let the sides (B C, F I) which are betwixt the equal angles, be supposed equal: In this case all the other parts are equal: as hath been already demonstrated in the *Scholium* of the fourth Proposition.

Again, suppose the sides A C, L I, which are opposed to the equal angles to be equal. Here, because the angles (B, C) are by the hypothesis equal to (F, I) the other angles, also (A, L) shall be equal by *Corollary* 9. *Proposition* 32. which Proposition depends not upon this. Therefore by the first part of this, all the other parts are equal.

Corollary. " Hence also, following *Thales*, we may " measure inaccessible distances. *e. g.* Let A D be an " inaccessible line; to which at the point A, let there be " erected the perpendicular A C. Let there be made the " angle (A C B) equal to the angle (A C D) the accessible " line A B shall be equal to the inaccessible A D. *Q. E. I.* Fig. 84.

PROP. XXVII. Theorem.

IF the right line G O shall cut two right lines which are parallel (A B, C F,) 1. The alternate angles (R L O, Q O L, likewise B L O, C O L) shall be equal. 2. The external angle G L B shall be equal to the internal one on the same side (that is, to L O F) as likewise G L R equal to L O C. 3. The two internal ones on the same Fig. 46.

ſame ſide (A L O, C O L) as taken together, ſhall be equal to two right ones, as likewiſe the two (B L O, F O L) equal to two right ones.

Fig. 46. The firſt part is thus proved. From O and L draw the perpendiculars O R, L Q. Theſe are perpendicular to the * two parallels A B, C F; and by definition 36, equal betwixt themſelves, they ſhall therefore (*a*) intercept equal parts of the parallels, and R L ſhall be equal to Q O. Therefore the triangles X and Z are equilateral to one another. Therefore (*b*) the alternate angles R L O, Q O L, which are oppoſite to the equal ſides R O, Q L, are equal. Which is the firſt thing. From whence it is likewiſe manifeſt, that the alternates B L O, C O L are equal. For becauſe, as well B L O, A L O, as C O L. F O L are equal (*c*) to two right ones; therefore B L O, A L O together, are equal to C O L, F O L. Therefore taking away the equals R L O, F O L, the remaining ones B L O, C O L, ſhall be likewiſe equal.

* *Per Axiom* 11.
(a) *Per Axiom* 12.
(b) *Per* 8. *l.* 1.
(c) *Per* 13. *l.* 1.

Part ſecond. The angle G L B is equal to that which is vertically oppoſite R L O (*d*.) But R L O, by the firſt part of this propoſition, is equal to L O F. Therefore GLB, the external angle, is equal to the internal remote one, which is on the ſame ſide, L O F.

(d) *Per* 15. *l.* 1.

Part third. A L O, by the firſt part, is equal to L O F. But LOF, with C O L, make (*e*) angles equal to two right ones. Therefore A L O, with C O L, doth the ſame.

(e) *Per* 13. *l.* 1.

Fig. 85. Corol. " Hence in imitation of *Eratoſthenes*, we learn " to meaſure the compaſs of the earth. For he obſerved, " that on the day of the ſummer ſolſtice, the ſun was " perpendicularly over *Siene*, a city of *Egypt*; and he " found by the means of a ſtile, perpendicularly erected, " that on the ſame day the ſun was diſtant from the ver- " tical point of *Alexandria*, a city of *Egypt*, ſituate almoſt " under the ſame region with the other, ſeven degrees, " with one fifth part of a degree; and he knew that " theſe two cities were about 5000 furlongs diſtant from " each other. From theſe things, by the help of this " propoſition, he determin'd the compaſs of the earth. " Let A be *Siene*, and B be *Alexandria*, where the Gnomon " B C is erected perpendicular to the horizon. Let DF " and E G be the ſolar rays parallel to one another as to

" ſenſe

"ſenſe. D A a ray perpendicular to the horizon of *Siene*; "and E G a ray oblique to the horizon of *Alexandria*, "and which paſſing by the top of the Gnomon, makes "with it the angle G C F, which is of $7\frac{1}{5}$ degrees. "Now ſeeing the angle G C F, is equal to the alternate "one A F B, and the meaſure of it is the arch A B of "$7\frac{1}{5}$ degrees; he found the compaſs of the earth by this "analogy, as $7\frac{1}{5}$ degrees are to 5000 furlongs; ſo the "whole circumference, which is of 360 degrees, is in a "groſs number [illegible]50000, the compaſs of the earth in "the ſame meaſure. Q. E. I.

PROP. XXVIII. Theorem.

IF *a right line* (*GO*) *cutting two right lines* (*A B, C F*) *makes the alternate angles* (*A L O, L O F*) *equal*; *the lines* (*A B, F C*) *are parallel.* Fig. 47.

If you deny it, let X L Z, paſſing through the point L, be parallel to C F. Therefore X L O (*a*) is equal to the alternate F O L, which cannot be, ſeeing by the hypotheſis A L O is equal to F O L. *(a) By the foregoing*

PROP. XXIX. Theorem.

IF *a right line* (*G O*) *cutting two right lines* (*A B, C F*) *ſhall make the external angle* (*G L B*) *equal to the internal oppoſite one* (*L O F,*) *or ſhall make the two internal angles on the ſame ſide* (*A L O, C O L*) *equal to two right angles*; (*A B, C F*) *are parallel lines.* Fig. 45, 46.

By the 15th of this book, G L B is equal to A L O, which is vertically oppoſite to it. But by the hypotheſis G L B is equal to L O F. Therefore alſo A L O is equal to its alternate one L O F. Therefore (*b*) A B, C F are parallel. *(b) By the foregoing*

Again, C O L with F O L make angles equal to two right ones. But by the hypotheſis C O L with A L O, make in all two right angles alſo. Therefore A L O, F O L,

F O L, the alternate angles are equal.' Therefore again, (a) A B. C F are parallel.

(a) *By the foregoing.*

Corollary: "From the second part of this proposition "it appears that every rectangle is a parallelogram.

PROP. XXX. Theorem.

Fig. 45. *IF two right lines (A B, C F) b[e] parallel to the same right line (D N) th[ey] are parallel betwixt themselves.*

It is manifest in itself, and from the foregoing propositions. For if all be cut by the right line G O, the external angle G L B is equal (*b*) to the internal opposite one L D N. Now L D N is an external angle in respect of D O F, and therefore (*c*) equal to it. Therefore also G L B is equal to L O F. Therefore A B, C F, (*d*) are parallel.

(b) *Per* 27. *l.* 1.
(c) *By the same*
(d) *By the foregoing.*

PROP. XXXI. Problem.

Fig. 48. *THROUGH a given point (A) to draw a parellel to a given right line (F C).*

From the point A, let there be drawn at random A L, cutting the given F C. At the point A, let there be made the angle (*e*) L A S equal to the angle A L F. The line A S will be parallel to C F; as is manifest from the 28th, the alternate angles SAL, A L F being equal.

(e) *Per* 23. *l.* 1.

As for the practice. Draw A L, and from the center L describe an arch I Q; and from the center A, with the same interval, describe the arch O X; from which, having taken off O B equal to I Q, the right line drawn through A and B will be the parallel fought. The demonstration depends upon the 29th, *l.* 1.

Fig. 49. Or otherwise thus. From a certain center P describe a circle which may pass through the given point A, and may cut the given line CF in Q and O. Take the arch O N equal to Q A. The right line A N shall be the parallel fought.

The demonstration hereof depends upon the 29th, *l.* 3 and the 28th of this.

PROP.

PROP. XXXII. Theorem.

PART I.

IN every triangle any one of the external angles (as F B C) is equal to the internal remote ones (A and C.) Fig. 51.

Through the point B draw (*a*) BL parallel to AC: because F A cuts the two parallels BL, A C, the external angle F B L ſhall be equal to the internal one A (*b*.) And because the line B C cuts the ſame parallels (B L, A C,) the angle L B C ſhall be (*c*) equal to its alternate one C. Therefore the whole angle F B C ſhall be equal to A and C both together *Q. E. D*

(a) *Per* 31. *l.* 1. (b) *Per* 27. *l.* 1. (c) *By the ſame.*

Corollaries.

1. THE external angle F B C is greater than either of the internal oppoſite ones A or C. Fig. 51.

2. Of the angles (C and A O B) having the ſame baſe, A B which is within, is the greater. Fig. 39.

For let A O be produced unto F, A O B, by this propoſition, is greater than O F B; and likewiſe O F B is by this greater than C. Therefore A O B is much greater than C. Fig. 55.

3. If from one point A there fall two right lines upon B C; one of them A O obliquely, the other A F perpendicularly; this laſt ſhall fall on the ſide of the acute angle A O B. For let it fall, if it may be, on the ſide of the obtuſe angle A O C; as for inſtance, in Q. In this caſe the acute angle A O B ſhall be external in reſpect of A Q B, and conſequently ſhall be greater than the right one, by *Corollary* 1, which is abſurd.

PROP.

PROP. XXXII. Theorem.

PART. II.

In every triangle the three angles taken together, are equal to two right ones, and therefore make 180 *degrees.*

Fig. 52. Draw forth one side A B unto F. The external angle F B C is equal *(a)* to the two internal opposite ones, A and C. But F B C with A B C, make *(b)* angles equal to two right ones. Therefore the two, A and C, with the same CBA, make angles equal to two right ones. *Q. E. D.*

(a) *By the first Part of this.*

(b) *Per* 13. *l.* 1.

Or thus, Draw the line H M parallel to A C, the alternate angles, as well O and A, as N and C, *(c)* are equal. But O, Q, N, make angles *(d)* equal to two right ones. Therefore also A, C, Q are equal to two right ones. *Q. E. D.*

Fig. 53.

(c) *Per* 27. *l.* 1.

(d) *Cor.* 1. *Prop.* 13. *l.* 1.

Corollaries.

4. THE three angles of any one triangle taken together, are equal to the three angles of any other triangle taken together.

5. If in a triangle one angle be right or obtuse, the rest are acute.

6. If in a triangle one angle be right, the two other angles together make one right angle.

7. In every triangle, the angle which is right, is equal to the other two taken together.

8. When you know of how many degrees one angle of a triangle is, you know at the same time how many degrees the two other angles, as taken together, do make up. And so on the contrary, when you know how many degrees two angles of a triangle taken together do make up, or what is the sum of them, you know at the same time of how many degrees the third angle is.

9. When two angles of one triangle, either severally or together, are equal to two angles of another triangle; the third angle of one triangle is also equal to the third of the other.

10. When two triangles have one equal angle, the sums also of the rest of the angles are equal.

11. When

11. When in an *Isosceles*, the angle contained by the equal sides is a right one, the two other are, each of them, half right angles. And the angles of an *Isosceles*, which are at the base, are always acute.

12. In an equilateral triangle, each angle is two thirds of a right angle. For it is one third of two right ones, therefore it is two thirds of one right one.

13. Hence a right angle (BAC) is easily divided into three equal parts: if upon AC be made the equilateral triangle Z; for seeing FAC is two thirds of one right one, BAF shall be one third of a right one. *Fig.* 54.

14. The perpendicular AF is the shortest of all lines which can be drawn from the point (A) unto some right line. For seeing the angle F is a right one, AOF shall by the 5th *Corollary* be an acute one. Therefore (*a*) AF is shorter than any other, as AO. *Fig.* 55. (a)*Per* 19. *l.* 1.

15. Only one perpendicular can fall from one point unto one right line. This is manifest out of the foregoing *Corollary*.

16. " Hence also we learn to determine the parallax of " the stars, or the difference of their true and apparent " place. Let A be the center of the earth, B the place " of the observer upon the surface of the same. Let " DBC be the angle of the star C, according to obser- " vation, or the *visible* angular distance thereof from the " vertical point; when in the mean while DAC is the " *true* angular distance. Now the external angle DBC, " which is given from observation, is equal to the angles " BAC and BCA, taken together; and consequently " the angle BCA is the difference of the angles DBC " and DAC. If therefore we shall from astronomical ta- " bles seek the angle DAC, or what at that time of ob- " servation is the true angular distance of the star from " the vertical point, when the angle DBC is at the same " time known by means of the Quadrant, the difference " of those angles BCA, which we call the *Parallax*, will " likewise be known. *Q. E. I.* *Fig.* 86.

Scholium.

BY the testimony of *Eudemus*, an ancient geometrician, *Pythagoras* was the finder out of this proposition, which indeed is a theorem most excellent in itself,

ſelf, moſt fruitful in its conſectaries, and of immenſe uſe in all parts of the mathematicks. *Ariſtotle* frequently makes mention of it, who alſo puts it for an example of the moſt perfect demonſtration. But like, as from this propoſition, we have already learned, how many right angles, the angles of a triangle are equivalent to; ſo by the help of the ſame, it will in the three following propoſitions be manifeſt, how many right angles, the angles of any rectilinear figure whatſoever, whether internal or external do make.

Theorem 1.

Fig. 56. IN every quadrangular figure, the four angles together make four right ones.

For if, through the oppoſite angles, you draw the right Line B F, this will cut the quadrangle into two triangles, without forming any new angles, whoſe angles together do (*a*) make four angles.

[a] *Per* 32. *l.* 1.

Theorem 2.

ALL the angles together of every right-lin'd figure make twice ſo many right ones, abating four, as are the ſides of the figure.

Fig. 57. From any point A within the figure, let there be drawn unto the angles of the figure right lines, which ſhall cut the figure into ſo many triangles as it hath ſides, and make no more angles, but thoſe of the center. Wherefore, when each of the triangles contains two right angles (*b*), they muſt all together contain twice ſo many right angles as there are ſides. Now the angles about the point A (*c*,) do make four right angles. Therefore, if from the angles of all the triangles, you take away the new angles which are about A, the remaining angles, which indeed do alone conſtitute the angles of the figure, will make twice ſo many right angles, excepting four, as are the ſides of the figure.

[b] *Per* 32. *l.* 1.

[c] *Cor.* 3. *Prop.* 13. *l.* 1.

Hence it appears, that all right-lin'd figures of the ſame ſpecies, or number of ſides and angles, have the ſum of their angles equal. Which thing is worthy of admiration.

The practice is thus: Double of the denominator of the figure, and from the product take away four; the remainder

der is the number of the right angles, which the internal angles of the figure do make.

Theorem 3.

ALL the external angles of any right-lin'd figure whatsoever taken together do make up four right angles. *Fig.* 58.

For each of the internal angles of the figure does (*d*) with its respective external one, make two right angles. Therefore all the internal ones, together with all the external ones, do make up twice so many right angles as are the sides of the figure. Now, by the preceding, the internal ones, together with four right angles added to them, make twice so many right angles as are the sides of the figure. Therefore the external angles are equal to four right ones. [d] *Per* 13. *l*, 1.

Wonderful truly is this property of right-lin'd figures; from whence it follows also, that all the right-lin'd figures of any species whatsoever have the sums of their external angles equal. And therefore the three external angles of a triangle are equal to the thousand external angles of a thousand-sided figure. Which observation is altogether worthy of admiration.

PROP. XXXIII. Theorem.

IF two right lines, which are equal and parallel, as (AB, CF) be joined by two others, (AC, BF;) these will also be equal and parallel. *Fig.* 59.

Let AF cut the parallels AB, CF. In the triangles QR, the alternate angles BAF, CFA (*a*) will be equal. Now the side AB is supposed equal to the side CF, and AF is common to both triangles. Therefore (*b*) the bases BF, AC are equal. (Which is the first part.) And then the angles at the bases AFB, FAC are equal; which being made by AF falling upon the right lines AC and BF, are alternate angles AFB, FAC equal. Therefore AC, BF are also (*c*) parallel. Which is the other part. [a] *Per* 27. *l*. 1. [b] *Per* 4. *l*. 1. [c] *Per* 21. *l*. 1.

Corollary.

Fig. 87. Corollary 1. "Hence we learn to meaſure as well "the heights of mountains above the horizon as their "horizontal lines. Let A B C the ſide of a mountain, "to which apply a great ſquare, or ſome inſtrument "equivalent thereto A D B. Then ſhall A D be equal to "H B, and D B equal to A H. Then coming unto the "lower part, which is from the point B unto the point C, "practiſe as before. So ſhall F B be equal to C F, and "E C be equal to B F. Which done, the ſides parallel "to the horizon, A D, B E, &c. added together will "give the horizontal line G C; and the perpendicular ſides "B D, E B, &c. added together, will give the height A G.

Fig. 59. Corollary 2. "Hence alſo we learn to eſtimate the "compoſition of motions. Let a body placed at A be dri- "ven in the ſame moment of time by the force A C, ac- "cording to the direction of the line A C, and by the "force A B, according to the direction of the line A B. "From the conjuction of theſe two forces it will de- "ſcribe the diagonal A F. For in this line of its motion, "neither of the forces is changed: for the body at F is "equally diſtant from both the lines of direction A C, "A B, as if it had been driven by either of the forces ſe- "parately; which thing can be ſaid of no other point. "And this corollary doth ſo fully agree agree with aſtro- "nomical and other mechanical phænomena, that it is "worthily reckoned by the famous Sir *Iſaac Newton*, as a "foundation of his geometrical philoſophy.

PROP. XXXIV. Theorem.

Fig. 89. *In every parallelogram the oppoſite ſides and angles are equal, and it is cut into two equal parts by the diameter.*

Becauſe A B, C F are (*a*) parallel, and A F falls upon them, the alternate angles B A F, C F A are (*b*) equal. Likewiſe becauſe A C, C F (*c*) are parallel, and upon them falls the line A F, the alternates C A F, B F A (*d*) are equal. Therefore the whole angle B A C is equal to the whole angle B F C. In the ſame manner B and C are ſhewed to be equal. Which was the firſt part.

Now becauſe it hath been already ſhewed, that the triangle Q, R, which have one common ſide A F, have

[a] *Per Def.* 35.
[b] *Per* 27 *l.* 1.
[c] *Per Def.* 35.
[d] *Per* 27. *l.* 1.

have also the angles adjacent to the common side equal, DAF to CFA; and CAF to BFA; the sides likewise shall be equal, *(a)* AB to FC, and BF to AC; and thus the whole triangles are equal. Which was the second part. (a) *Per* 26. *l.* 1.

Scholium.

PARTLY from this theorem, and partly from a definition to be premised to the second book, the measuring of a right-angled parallelogram is easily deduced. The area thereof being produced by the multiplication of the two contiguous sides AF, AC one by another. *e.g.* Let AF be a line of 8, AC a line of 4 feet. Multiply 8 by 4, there arises 32 square feet for the area of the rectangle. *Fig.* 60.

But the area of a square is had from the multiplication of the side FI by itself; as if FI be of 5 feet, multiply 5 feet into itself, there will arise 25 square feet for the area of the square. *Fig.* 61.

The demonstration is manifest from this proposition, if parallel lines be drawn through the divisions of the sides.

Corollary. " Hence surveyors do easily divide the area " of a field, when it is a parallelogram. For let ABCD " be the parallelogram field: AD the diameter, or dia- " gonal line of the same, the middle point whereof is " marked F. Whatsoever right line, as EG, passeth " through the point F, it divides the field into equal " parts EACG, EBDG. For the triangle ABD is " equal to the triangle ACD, and * the triangle AEF " equal to the triangle GFD. If therefore to the tra- " pezium EBDF, instead of the triangle AEF, you " shall add the triangle which is equal to it, GFD, you " will not change the area; but the trapezium EBDG " will be equal to the triangle ABD, or to half the pa- " rallelogram, and consequently to the trapezium AEGC. " *Q. E. D.* *Fig.* 88. * *Per* 26. *l.* 1.

PROP. XXXV, XXXVI. Theorems.

Fig. 62.

PArallelograms upon the ſame or equal baſes (AB) and between the ſame parallels (CQ, AX) are equal.

Becauſe AL, BQ (*a*) are parallel, and CQ cuts them, the external angle CLA ſhall (*b*) be equal to the internal one FQB. Then becauſe, as well CF as LQ are equal (*c*) to the ſame AB, CF is equal to LQ. Add then FL to both, the whole lines CL, FQ are equal. Moreover AL, BQ are equal (*d*.) Therefore the triangles CLA, FQB (*e*) are equal. Therefore taking away the common triangle FOL, the planes FOAC, QBOL remain equal: To each of which trapeziums add the triangle AOB, the whole parallelograms ACFB, ALQB become equal. *Q. E. D.*

(a) *Per Def.* 35. (b) *Per* 27. *l.* 1. (c) *Per* 34. *l.* 1. (d) *Per* 34. *l.* 1. (e) *Per* 4. *l.* 1.

This propoſition will be made univerſal. *Prop.* 1. *l.* 6. beginners may here obſerve, that although of two parallelograms which are between the ſame parallels infinitely produced and upon the ſame baſe, one of them be extended unto an infinite length; it ſtill remains but equal to the other, by the force of the preſent demonſtration.

[" From hence it follows, that two cities in magnitude " equal, may ſo much differ in compaſs, that the circumference of one may exceed that of the other an " hundred or a thouſand times. If, for inſtance, one be " of a ſquare figure or rectangular; but the other a parallelogram, betwixt the ſame parallels indeed with the " former, but very oblong.

" Moreover, it hence follows, that figures of equal " compaſs round may contain areas vaſtly different.]

Scholium.

Fig. 62.

FROM this theorem we may learn to meaſure any parallelogram. For the area of it is produced from the perpendicular altitude QX, or CA multiplied into the baſe AB.

For

For the area of the rectangle CB which is equal to that of the parallelogram BL is made (a) by AC, multiplying AB. Therefore, &c. (a) *By the foregoing. Scholium.*

PROP. XXXVII, XXXVIII. Theorems.

TRiangles (ACB, ALB) upon the same, or equal bases (A B,) and between the same parallels (CI, AZ) are equals. *Fig.* 63.

Draw the lines BF, BI parallel to the sides AC, AL. The parallelograms ACFB, ALIB *(b)* are equal. But the given triangles are halves of those parallelograms *(c,)* Therefore the given triangles *(d)* are equal. (b) *By the foregoing.* (c) *Per* 34. *l.* 1. (d) *Per Axiom* 6.

This proposition will be made universal, *Prop.* 1. *l.* 1. Let beginners mark the same thing here concerning triangles, which we bid them to note in the foregoing proposition concerning parallelograms.

Corollary (1.) " Hence surveyors easily divide the " area of a triangular field. Let ABC be the field, and " let the base BC be bisected in D. The triangles ABD, " ADC upon the equal bases BD and DC, and having " a common top A, or being between the same parallels, " are equal. *Q. E. F.* *Fig.* 89.

Corollary (2.) " Hence we also gather, with the famous Sir *Isaac Newton*, that the areas which all bodies " whatsoever that revolve round about an immoveable " center, towards which they are impell'd, do describe, " are both in immoveable planes, and are proportional to " the times of description. For let the time be divided " into equal parts, and in the first equal part of time, " let the body, by the impress'd force, describe the right " line AB. The same body, in the second part of time, " if nothing hindered, would go forward strait unto c, de- " scribing the line Bc, equal to AB; so that the areas " made by lines drawn from the center ASB, BSc *(a)* " would be equal. But when the body comes unto B, let the " force act with one single impulse, but a great one, and " make the body to deflect from Bc, and to go forwards " in the right line BC: *i. e.* let the centripetal force be in " that place, to the force before impuls'd, as Cc or Bg *Fig.* 90. (a) *Per* 37. *l.* 1.

 " is

(b) *Per Corol.* 2: *Prop.* 33. *l.* 1.

(c) *Per* 37. *l.* 1.

(d) *Per Axiom* 1.

"is to Bc; in this case the body will (*b*) describe the "diagonal BC. Let there be drawn parallel to BS, the "right line Cc meeting BC in C. In the second part of "time completed, the body will be found in the point C; "in the same plane with the first triangle SAB. Join "SC. The area made by a ray drawn from the center, "that is, the triangle SBC will be equal to (*c*) SBc, and "consequently to the first triangle SAB (*d*) By the "same argument, the body, in the third equal part of "time, would, by its present force, reach from C unto "d, so that the line Cd should be equal to the line Bc "or AB. But if the centripetal force, whether it be "greater or less, does again act upon it in the point "C, in the end of the third part of time, it will be found "somewhere in the line Dd, parallel to SC; and there-"fore, as before, supposing the said force to be equal or "unequal to what it was before, it will be found to have "described the diagonal CD, and will be found in the "point D; and a ray being drawn from the center, the "triangle SDC will be equal to that SdC, and con-"sequently to the others SCB, SAB, which are equal "one to the other. In like manner, if the centripetal force "act successively in the points D, E, F, and be the cause "that the body, in the several parts of time respect-"ively, describes the diagonals DE, EF, &c. the areas "now made, as afore, will in the same plane, and tri-"angles will be described equal to the former triangles. "Therefore in equal times, equal areas are described in "an immoveable plane; and so the sums of the areas "SADS, SAFS will be amongst themselves as the "times wherein they were described. Now let the num-"ber of the triangles be increased, and their wideness di-"minished infinitely, both that last perimeter of them, "ABCDEF, will be a curve line, and the areas de-"scribed in one and the same immoveable plane, "will in this case also be proportional to the times as well "as before. *Q. E. D.*

PROP.

PROP. XXXIX. XL. Theorems.

EQUAL triangles (ACB, AFB) upon the same, or an equal base (AB) and on the same side, are between the same parallels (AB, CF). Fig. 64.

If you deny it, let CL be parallel to AB, and let BL be drawn. Then, ALB is equal to ACB (*a*) But by the hypothesis, AFB is equal to ACB. Therefore ALB and AFB are equal; *i. e.* a part is equal to the whole. Which cannot be. Therefore, *&c.*

(*a*) By the foregoing.

[Corollary (1.) "Hence also, with the famous Sir "*Isaac Newton*, we gather, that all bodies which are "moved in curve lines, and describe areas about some "center proportional to the times, are perpetually urg'd, "and press'd by a force impelling towards the center. "For because of the equality of the triangles SCB, ScB "described upon the same base SB, the points C and c "shall be in a line Cc, which is parallel to the base; and "so the figure BcCg shall be a parallelogram; the sides "whereof Bc and Bg are * the lines of the directions of "the forces, and BC is the diagonal. The body there-"fore is urged unto C by the force Bg, which tends unto "S the center. And so in all the points, C, D, E, F. "*Q. E. D.*

* Per Corol. 2. Prop. 33. l. 1.

Corollary (2.) "Seeing therefore in the motion of "the primary planets, the areas made by rays, or right "lines drawn from them unto the sun, are always pro-"portional to the times, as all astronomers know, the "planets are urged by a perpetual force, which tends to "the sun. And the same thing is equally true of the se-"condary planets, with respect to their primary ones.]

PROP. XLI. Theorem.

Fig. 65. *IF a triangle (A F B) be in the same parallels with a parallelogram (A L) and have the same, or an equal base (A B) it is half of the parallelogram.*

(a) *Per* 37. 38. *l.* 1. (b) *Per* 34. *l.* 1. Draw C B. The triangles A F B, A C B are (*a*) equal. But A C B is half of the parallelogram A L (*b*.) Therefore A F B also is half of A L. *Q. E. D.*

Scholium.

Fig. 65. FROM this proposition, with the scholium of *Prop.* 35. we learn, that the area of whatsoever triangle, as A F B, is produced from half the altitude F I multiplied into the base A B, or half the base multiplied into the altitude. Wherefore one side of a triangle being known, and the height, that is, the perpendicular which falls upon the known side from the opposite angle, the measure of the triangle is given. As if the base A B be of an 100 feet, the height F I, 85, multiply half the base, 50 by 85, and you have the area of the triangle A F B=4250 feet square. Further, the altitude of a triangle, when the area of it is in all points accessible, may be known mechanically as well as the sides. But if the area of it cannot be gone over, the height may be found geometrically by 12 and 13. *L b.* 2. as we shall there shew.

In a rectangled triangle, the height is the same with either of the sides about the right angle. Half of this therefore multiplied into the other side adjacent to the right angle, will give the area of the triangle.

PROP. XLII. Problem.

Fig. 66. *TO make a parallelogram with an angle equal to a given one (O,) and equal to a given triangle (ACB.)*

(a) *Per* 31. *l.* 1. (b) *Per* 23. *l.* 1. (c) *Per* 31. *l.* 1. Bisect the base A B in F. Through C draw C E parallel (*a*) to A B. Make the angle B A L equal to the given one O (*b*.) Draw FI parallel (*c*) to AL. AL, IF shall be that which was sought for.

For

For let FC be drawn. The parallelogram A I hath an angle L A F equal to the given one O, and is equal to the given triangle A C B; seeing that, as well the triangle A C B (*d*) as the parallelogram A I (*e*) is double to the same triangle A C F. (d) *Per* 38. *l.* 1. (e) *By the foregoing.*

Corollary.

THE triangle A C B being given, a rectangle equal to it is had, if there be drawn a line parallel to the side A B; and A B being bisected in F, the perpendicular B Q be erected. For the rectangle under F B and Q B will be equal to the triangle A C B. *Fig.* 66.

PROP. XLIII. Theorem.

IN a parallelogram (as B L) the complements (BQ, OL) of those parallelograms which are about the diameter (RF, CS) are equal. *Fig.* 67.

If through any point of the diameter A Q, as the point O, C F be drawn parallel to the side A B, and R S parallel to the side B Q; the whole parallelogram B L is divided into four parallelograms, whereof two are about the diameter R F, C S, the other two, B O, O L, are the complements of these unto the whole parallelogram B L.

Their equality is thus proved. The triangles (*f*) A B Q, A L Q are equal. Likewise the triangles A R O, O C Q (*g*) are equal to the triangles A F O, O S Q. Therefore, if from the equals (*h*) A B Q. A L Q, you take away equals, on this side A R O, O C Q, on that A F O, O S Q; then B O and O L shall remain equal. *Q. E. D.* (f) *Per* 34. *l.* 1. (g) *By the same.* (h) *Per Axiom* 3.

PROP. XLIV. Problem.

UPON a given right line (O S) to constitute a parallelogram, in a given angle (X,) which parallelogram shall be equal to a given triangle (V.) *Fig.* 68.

Make a parallelogram (*a*) R C equal to the given V, having its angle R O C, equal to the given one X, and (a) *Per* 42. *l.* 1.

join the side R O directly to the given line O S, so as to make one right line therewith. Then through S draw
(b) *Per* 31. *l.* 1. SQ (*b*) parallel to OC, which SQ, let BC meet when it is produced unto Q. Then let a right line, drawn through Q and O, meet B R produced unto A. Which done, through A draw A L, parallel to O S, which A L, let C O and Q S meet, when it is produced unto F and L; the parallelogram O L is that which was required.

(c) *By the foregoing.* For O L (*c*) is equal to R C, that is, by the construction, to the given triangle V, and is at the given line O S; and
(d) *Per* 15. *l.* 1. (*d*) the angle F O S is equal to the angle R O C; that is, by the construction, equal to the given angle X.

Scholium. " This proposition contains a certain geo- " metrical division. For in the vulgar arithmetical di- " vision, the number to be divided may justly be considered
Fig. 91. " as being a certain rectangle. *e. g.* Let the rectangle " AB comprehending 12 square feet, be to be divided by " 2; *i. e.* a rectangle is to be found equal to that A B " of 12 square feet, one of whose sides shall be only 2 " feet: From whence it comes to be enquired of what " number the side sought shall consist; which side is to " be esteemed a certain quotient of this division. Which " thing is performed geometrically after this manner. " With a pair of compasses take the line B D of 2 feet, " and draw the diagonal D E F. The line A F is that " which is sought for. For the complements E G and
* *Per* 42. *l.* 1. " E C are * equal; and in the rectangle E G, one side, " E H, is equal to the line B D, which is of 2 feet; " and the side E I, is equal to A F.

" This kind of division is called *Application*, because the " rectangular Space A B, is *applied* to the line B D or " EH; and hence it comes, that Division is often named " *Application*; respect being had to the practice of the old " geometricians, who always made more use of geome- " trical construction, which requires only a rule and a " pair of compasses, than of arithmetical computation, " which is performed by numbers.

PROP.

PROP. XLV. Problem.

UPON a given line (IQ) and in a given angle (H) to make a parallelogram equal to a given rectilineal figure (CBA). *Fig.* 69.

Resolve the given rectilinear into the triangles A, B, C, by drawing the right lines F L, F I.

Upon the given line I Q in the given angle H, make (*a*) the parallelogram I V equal to the triangle A. Then the right line I R being produced infinitely towards P; upon the right line R V, in the angle V R P, (*b*) make the parallelogram R Z equal to the triangle B. Again, upon the line S Z, with the angle Z S P, make the parallelogram S G equal to the triangle C. This done, I say, I G is the parallelogram sought for.

(a) *Per* 44. *l.* 1.
(b) *By the same.*

For (*c*) the angle Z V R is equal to its alternate I R V. But (*d*) Q V R and I R V, are equal to two right angles. Therefore also Q V R and Z V R, are equal to two right ones. Therefore * Q V and Z V fall directly so as to make one line right. After the same manner I might shew that Q Z and Z G make one right line. Therefore the whole Q V Z G is one right line, and is also parallel to I X, seeing by the construction Q V is parallel to I P. Now X G also (*e*) is parallel to I Q. Seeing X G is parallel to S Z, and S Z to R V, and R V to I Q.

(c) *Per* 27. *l.* 1.
(d) *By the same.*
* *Per* 14. *l.* 1.
(e) *Per* 30. *l.* 1.

IG therefore (*f*) is a parallelogram; but that it is such an one as was required, is manifest from the construction.

(f) *Per Def.* 35.

[Corollary. " Hence is easily found the excess whereby
" a greater rectilinear figure exceeds a lesser: To wit, if
" unto the same right line I Q be applied parallelograms
" respectively equal to the two right-lin'd figures. For
" that parallelogram, by which the greater rectilinear ex-
" ceeds the lesser, will give the difference of them. *Q. E. I.*]

Scholium.

WE will here add a problem that will be useful for the [illegible] of proposition 14. *l.* 2.

A quadrangular figure, B F, being given, to describe an equal rectangle. *Fig.* 70.

Resolve

Resolve it into triangles by the right line A C. From the opposite angles, let down the perpendiculars B O, F I. Bisect A C in S. From S erect the perpendicular S L, equal to the two, B O, F I, put together. The rectangle, comprehended under L S and S C, is equal to the given B F. The demonstration appears out of proposition 41.

PROP. XLVI. Problem.

Fig. 71. *FROM a given right line (AB) to describe a square.*

Erect two perpendiculars equal to the given A B; to wit,
(g) *By the* A C, B E, then join C E. I say the thing is done.
Construction For seeing the two angles A and B are (*g*) right ones,
A C and B E shall (*h*) be parallel; but they are also (*a*)
(h) *Per* 29. *l.* 1. equal. Therefore C E and AB are (*b*) parallel and equal.
Therefore the figure is parallelogramical and equilateral.
(a) *By the* But all the angles also are right ones (for seeing A and B
Construction. are right angles, the opposite ones (*c*) E and C are right also.) Therefore the figure AE is a square.
(b) *Per* 33. *l.* 1.

[" In the same manner you may easily describe a rect-
(c) *Per* 34. *l.* 1. " angle, which hath the two unequal sides given.]

PROP. XVLII. Theorem.

IN every right-angled triangle (as A B C) the square of the side (AC) which is opposite to
Fig. 72. *the right angle, is equal to the two squares together of the two other sides (A B.)*

Let I C and B F be drawn; and B E parallel to A F. Now if to the right, and therefore equal angles I A B, F A C, there be added the common angle B A C, the
(d) *Per* wholes, I A C and F A B shall be equal. But in the tri-
Def. Squa. angle, I A C, F A B, the sides which contain those equal
(e) *Per* 4. angles, are equal (*d*) amongst themselves, to wit, I A,
l. 1. C A, to B A, A F, each to each. Therefore the triangles I A C, F A B, (*e*) are equal. Which, because they stand upon

upon the ſame baſes, I A, F A, with the parallelograms, A B L I and Z A F E, and between the ſame parallels, I A, L B C, and A F, E Z B, they are halves (f) of thoſe parallelograms. Therefore the parallelograms, A B L I, Z A F E, as being doubles of equals, are equal betwixt themſelves. By the ſame reaſoning, if right lines, A X, BR, were drawn, it might be ſhewn that the parallelograms EC, B X, are equal. Therefore the whole, A R is equal to I B and B X, together. *Q. E. D.* (f) *Per* 41. *l.* 1.

It was taken for granted that L B C is parallel to I A, in order to which L B and B C muſt be one right line. Now that they are ſo, is manifeſt from the 14th, ſeeing the angles L B A and C B A, are both right ones by the hypotheſis.

Scholium.

THIS theorem (which, *Prob.* 31. *l.* 6, *Euclid* extends unto all like or ſimilar figures) is commonly call'd the *Pythagoric* theorem, from *Pythagoras*, the inventor of it; who, as is atteſted by *Proclus*, *Vitruvius* and others, offer'd ſacrifices to the muſes, as ſuppoſing himſelf to have been helped by them in ſo excellent an invention; in which thing he ſhewed himſelf to be ignorant of God, the Lord of ſciences, the true and only author of all wiſdom; or certainly, if he knew him, he glorified him not as God. There is frequent and notable uſe of this theorem through all the mathematics; and in particular, it opens a way unto the knowledge of incommenſurable magnitudes, a main ſecret of geometrical philoſophy.

That the ſide of a ſquare is incommenſurable to the diameter, is a thing much celebrated amongſt the old philoſophers, *Ariſtotle* and *Plato* eſpecially; inſomuch that *Plato* would ſay, that he who knows not this, is not a man, but a beaſt. Now the knowledge of this myſtery ſeems to have taken its riſe out of this 47th propoſition. For ſeeing in the ſquare A E, the angle A is a right angle, the ſquare of the diameter C B ſhall be equal to both the ſquares of the ſides, A B, A C, and therefore double to one of them. Wherefore ſeeing the ſquare of C B is 2, and the ſquare of the ſide A B is 1, or unity, the diameter C B ſhall be the ſquare root of 2, and the ſide A B the ſquare root of unity, itſelf; the *ratio* of which quantities *Fig.* 71.

quantities (as it will be demonstrated in its place) cannot be explicated in numbers, and therefore they are incommensurable.

And by this one argument alone, if all others were wanting, it might evidently be made out, that geometrical magnitudes cannot be made up of a definite number of points: for otherwise none would be incommensurable; forasmuch as a point would be the common measure of all.

To these things we will subjoin three problems, which are deduced out of the present proposition, and are of frequent use.

Problem 1.

Fig. 73. IF any number of squares are given, to make one equal to them altogether.

Let there be three or more squares given, whose sides are A B, B C, C E. Make the right angle F B Z, having indefinite sides, and unto the sides of it transfer A B and B C, and then join A C. The square of A C shall be equal to the squares of A B and B C together (*a.*) Then transfer A C from B unto X, and CE the third given side, transfer from B unto E, and join E X; the square of E X shall be equal (*b*) to the squares of E B (or E C) and B X together; that is, equal to the three given squares, whose sides are A B, B C, C E: And so on as long as you please.

(a) *Per* 47. *l.* 1.

(b) *By the same.*

Problem 2.

Fig. 74. TWO unequal right lines being given (A B, B C) to determine that square, whereby the square of the greater (A B) exceeds the square of the less (B C.)

From the center B, with the interval A B, describe a circle. Then from C erect a perpendicular C E, cutting the circumference in E. The square of C E is the excess or difference which is sought for.

For let E B be drawn. The square of B E, that is, of A B is equal to the squares (*a*) of B C and C E together. Therefore, &c.

(a) *Per* 47. *l.* 1.

Problem 3.

Fig. 75.

ANY two sides of a right-angled triangle being known to find the third.

Let

Let the ſides containing the right angle be A B, A C, the one of 6 feet the other of 8. You are to find of how many feet the ſide C B, which is oppoſite to the right angle, is. To do which, multiply 6 and 8 each of them by itſelf. From which multiplication there will ariſe for the ſquares of thoſe two ſides 36 and 64; the ſum of which is 100. The ſquare root of 100, which is 10, gives the feet of the ſide B C, whoſe quantity was ſought. This demonſtration offers itſelf in and from this 47th propoſition, for the ſum of the ſquares B A and C A is equal to the ſquare of B C. Therefore the root of the ſum of them is equal to the root or ſide B C.

Then let the ſides A B, B C be known, the one of 6 feet, the other of 10, you are now to find A C. Take the ſquare of the ſide A B which is 36, out of the ſquare of the ſide B C = 100. The remainder 64 ſhall be the ſquare of the ſide A C. The root therefore of 64, which is 8, gives the feet of the ſide A C.

Corollary. "From hence we derive the original of *Fig.* 92.
"the tables of ſines, tangents and ſecants. For, in-
"ſtance, let A C the ſemidiameter of the circle be of
"100,000 parts, and the angle B A D of 30 degrees. * *Per Co.* 1.
"Becauſe the chord or ſubtenſe of 60 degrees is * equal *Prop.* 15. *l.*
"to A C the ſemi-diameter; B D the ſine of 30 degrees *l.* 4. *and Co.*
"ſhall be equal to half the ſemi-diameter, or ½ AC; and 2. *Prop.* 3.
"therefore ſhall contain 50,000 parts. But now in the *l.* 3.
"right-angled triangle A D B, the ſquare of A B is equal
"to the ſquares of A D and B D. Wherefore let the
"ſemi-diameter A B be ſquared (by multiplying 100,000
"by 100,000) and from that ſquare ſubtract the ſquare of
"B D; the remainder ſhall be the ſquare of A D, or
"of the coſine equal to it B F; out of which extract the
"ſquare root, and you will have the line B F or A D.
"Then by this following analogy, A B : B D : : A E :
"C E, or A D : B D : : A C : C E, will be had the tan-
"gent C E. And then laſtly, if the ſquare of A C be
"added to the ſquare of C E, the root of the ſum being
"extracted will be the ſecant A E. *Q. E. I.*

PROP.

PROP. XLVIII. Theorem.

Fig. 76. *If in a triangle the square of one of the sides (AB) be equal to the two squares of the other sides (AC, BC) taken together, the angle (ACB) which the two other sides contain, is a right angle.*

If not, the angle ACB will be greater or less than a right angle. In either of which cases (as it will be demonstrated, *Prop.* 12, 13, *l.* 2. which propositions depend not on this) the square of AB will not be equal to the squares of AC, BC together; which is contrary to the hypothesis.

Or thus. Draw FC perpendicular to AC, and equal to CB, and join AF. The square of AF is (*a*) equal to the squares of FC, CA together; that is, (*b*) to the squares of BC, CA; that is, by the hypothesis, to the square of AB. Therefore the right lines AF, AB are equal. Because therefore the triangles X and Z are mutually equilateral, the angles at C (*c*) are equal. Therefore they are both right angles (*d*). *Q. E. D.*

(*a*) *Per* 47. *l.* 1.
(b) *By the Construction.*
(c) *Per* 8. *l.* 1.
(d) *Per Def.* 14.

THE

THE

Elements of EUCLID.

BOOK II.

THIS book is ſmall in bulk, but great indeed in the excellency and uſefulneſs of its theorems. Young beginners will not, mark what I ſay, be at firſt able to diſcover it; but being further advanced, they will, from their own experience, and with the greateſt certainty, apprehend that it is moſt true.

A DEFINITION.

A right-angled parallelogram (as A E) which is wont ſimply, and without any addition, to be call'd a rectangle) is ſaid to be contain'd under the two lines (A C, A F) which determine the magnitude of it. *Fig.* 60. *l.* 1.

For the one of them A C determines the height, the other A F the breadth of it. Now, if the ſide A C be underſtood to be carried perpendicularly along the whole A F, or A F along A C, by that motion the rectangle or its area will be produced. Wherefore a rectangle is rightly ſaid to be produced from the drawing of two lines into one another, or the multiplication of them one by the other. When therefore you have theſe words [the rectangle under (or of) A C, C B,] or for brevity's ſake, [the rectangle A C B] there is meant that rectangle which is contained under A C and C B, multiply'd one into the other. In like manner, when we ſay the rectangle under A B, B C, or the rectangle A B C, there is deſign'd the rectangle contained *Fig.* 2. *l.* 2.

tain'd under the right lines AB and BC, multiply'd one another.

Moreover, of rectangles some are oblong, some are square. The oblong rectangle is that which hath its contiguous sides unequal, or which is contained under two unequal right lines. The square rectangle, that which is contain'd under two equal right lines.

PROPOSITION I. Theorem.

Fig. 1. l. 2. *If there be two right lines (AB, AC) one whereof is divided into as many parts as you will (AE, EF, FC;) the rectangle compriz'd under those two (AB, AC) is equal to all the rectangles together, which are contain'd under the undivided line (AB) and the several parts of the divided line (AE, EF, FC.)*

Make AB perpendicular to AC, thro' B draw the infinite line BR parallel to AC. From E, F, C, erect the perpendiculars EI, FL, CQ. BC will be a rectangle under AB and AC; and is equal to the rectangles BE, IF, LC; that is, (because as well IE as LF are equal *Per 29, & 34. l. 1.* * to AB) equal to the rectangles under AB, AE; AB, EF; AB, CF.

Scholium.

THE ten first theorems of this book are true also in numbers, if they as lines be divided into parts. The numerical rectangles are produced from the multiplication of two numbers, and the numerical squares from the multiplication of the same number by itself.

["Let the undivided number be 9, and the divided " one 12. The rectangle which is from 9 multiplying " 12=108 will be equal to the three rectangles, 27, 36, " and 45, which are produced from 9 multiplied by 3, and " 4 and 5, respectively and separately. Or let the num- " ber 432 be as it were a multiplicand divided into 400 " and 30 and 2; and the number 8 an undivided multi- " plier; 8×432=3456 will be equal to 8×400=3200+ " 8×30=240+8×2=16. And from this proposition " therefore the demonstration of multiplication is to be " deriv'd.]

PROP.

PROP. II. Theorem.

If the right line (A B) be cut any where (as in C,) the two rectangles under the whole (A B) and the parts (AC, C B) are equal to the square of the whole line (A B.) Fig. 2.

[" For AD is the ſquare of the whole, and AH, CD are " rectangles under the whole A B, and the parts A C, C B.] Fig. 17.

[" Let the number 8 be divided into 5 and 3; the " ſquare of the whole 8x8=64, is equal to the rectan- " gles 8x3=24, +8x5=40.]

PROP. III. Theorem.

LET a right line, as (AB) be cut any where, (as for inſtance in C, the rectangle contain'd under the whole A B, and either of the parts, (BC) is equal to the rectangle under the parts (AC, C B) together with the ſquare of the ſaid part (B C.) Fig. 3.

[" For A F is the rectangle under the whole line A B, " and the part A C; and C F is the rectangle under the " parts, as A E is the ſquare of the part A C.] Fig. 18.

" In numbers. Let the number 7 be divided into the " parts 3 and 4. The rectangle of 7x3=21 is equal to " the rectangle of 3x4=12, together with the ſquare " 3x3=9. In like manner 7x4=28, is equal to the " rectangle 3x4=12+ the ſquare 4x4=16.]

PROP. IV. Theorem.

LET a right line, as (F L) be cut any where, as in (O,) the ſquare of the whole ſhall be equal to the ſquares of the parts (FO, O L) and to two rectangles contain'd under the parts (FO, OL.) Fig. 4.

[" For F D is the ſquare of the whole, and C G and " C L the ſquare of the parts; and C F, C D, two rect- " angles under the parts.] Fig. 19.

" In numbers. Let the number 10 be divided into " two parts, 7 and 3. The ſquare of $10 \times 10 = 100$ is " equal to the ſquares of the parts $7 \times 7 = 49$, and $3 \times 3 = 9$, " and to the two rectangles $7 \times 3 = 21$, and $7 \times 3 = 21$. " And on this propoſition depends the extraction of the " ſquare root.

Fig. 19. Corollary (1.) " Hence it is manifeſt, that the parallelo- " grams about the diameter of a ſquare, (O I, H K) are " ſquares.

(2.) " As likewiſe, that the diameter of every ſquare " biſects the angles of it.

(3.) " And that the ſquare of half the line is a fourth " part of the ſquare of the whole line. For in that caſe " the rectangles and ſquares end in four equal ſquares.

PROP. V. Theorem.

Fig. 5. *IF a right line, as (Q X) be cut equally in (R) and unequally in (S,) the rectangle contain'd under the unequal parts (Q S, S X) taken together with the ſquare of the intermediate part (R S) ſhall be equal to the ſquare of the half (Q R.)*

Fig. 20. [" For Q H is the rectangle under the unequal parts " and L G the ſquare of the intermediate part, and R F " the ſquare of half the line; and therefore, becauſe the " rectangle Q L is equal to the rectangle S F, and the reſt " of the ſpace is common to both, the propoſition is " manifeſt.]

[" Let the number 8 be divided equally, that is, into " 4 and 4, and unequally into 5 and 3. The rectangle " of $5 \times 3 = 15$ together with the ſquare $1 \times 1 = 1$ ſhall be " equal to the ſquare $4 \times 4 = 16$.]

PROP. VI. Theorem.

Fig. 6. *IF a right line (A B) be divided into two equal parts in C, and to it a certain right line (B F) be adjoin'd; the rectangle contain'd under the whole compound line (A F) and the adjoin'd one (B F) taken together with the ſquare*

of half the line (C B) ſhall be equal to the ſquare of (C F) which is compounded of half the line (A B) and the adjoin'd one.

[" For A N is the rectangle under the whole compound " line and the adjoin'd one; and G K the ſquare of half " the line A B; and C E the ſquare of the line com- " pounded of half the line A B, and that which was ad- " ded. Wherefore, becauſe the rectangle H E is equal to " the rectangle A K, and the reſt of the ſpace is common " to both, A N and K G is equal to C E. *Q. E. D.*] Fig. 21.

[" If the number 6 be divided into the two equal parts, " 3 and 3; and to it be added the number 2; the rect- " angle of 8×2=16, taken together with the ſquare 3×3 " =9, ſhall be equal to the ſquare 5×5=25.]

" Corollary. " Hence, with *Maurolicus*, with one ſingle " obſervation, we learn to meaſure the diameter of the " earth. Let the altitude of the mountain A D be known, Fig. 22. " and A B the line touching the earth be known by " meaſuring. Let the line D E be cut into two equal " parts in the center C, and to it be added the line " A D. Now, becauſe the rectangle under A E, A D, " together with the ſquare of D C, is by this propoſition " equal to the ſquare of A C, that is, equal to the * ſquares* Per 17. " of the lines A B, B C: From hence it follows, that if *l.* 1. " you take away on both ſides the ſquare of C D or C B, " the rectangle which is under A E, A D is equal to the " ſquare of A B. Therefore let the known ſquare of A B " be divided by the known altitude of the mountain A D, " and the quotient will give the line A E. From which " ſubtract the known altitude of the mountain A D, the " remaining line D E will be the diameter of the earth. *Q. E. I.*

PROP. VII. Theorem.

Fig. 7.

IF a right line (A B) be cut any where, (as in C,) the ſquare of the whole line (A B) taken together with the ſquare of either of the ſegments (A C) is equal to two rectangles contained under the whole (A B) and that ſegment (A C,) together with the ſquare of the other ſegment (C B.)

Fig. 23. [" For E B is the ſquare of the whole line, and A L " the ſquare of the part A C. But the two rectangles " under the whole line, and that part E I, H L, together " with G B, the ſquare of the other part, poſſeſs the " ſame ſpace that E B and the ſquare of A C doth. " Therefore they are equal to E B and the ſquare of A C.

" Let the number 13 be divided into any two parts, " as 9 and 4. The ſquare 13x13=169, together with " that 9x9=81, is equal to 13x9=117, and 13x9=117, " and the ſquare 4x4=16.]

PROP. VIII. Theorem.

Fig. 8. *IF a right line (LF) be divided into two equal parts in (I,) and to it a certain right line be adjoin'd (F O;) the rectangle (L I O,) which is contain'd under the half of the line (L I) and the line (IO) that is compounded of half the aforeſaid line, and the line adjoin'd, this rectangle taken four times, together with the ſquare of the adjoin'd line (F O,) ſhall be equal to the ſquare of the whole compound line (L O.)*

Fig. 24. [" For A L is the ſquare of the whole compound, con- " taining four equal rectangles under L I and I O (to wit, " D R, B Q, R O, and the fourth made up of L R and " Q H added together) and with thoſe four rectangles the " ſquare H E. From whence the propoſition is manifeſt.

" Let the number 12 be divided into 6 and 6; and the " number 4 be added to it. The four rectangles, 10x6= " 240 and 4x4=16 are equal to the ſquare 16x16=256]

PROP. IX. Theorem.

Fig. 9. *IF a right line (AC) be divided equally in (B) and unequally in (F,) the ſquares of the unequal parts (AF, FC) will be double to the ſquares of half the line (A B,) and of the intermediate part (B F.)*

[" Le-

[" Let B E be equal and perpendicular to B A. From " hence the construction being made, as the figure shews, " the lines A B, B E, B C, will be equal: As also the lines, " A G, G Q will be equal. The angles A E C, A B E, " G E, E G Q, Q F C will be right; and the angles " A E B, B E C, E C A, C Q F, E Q G half right ones. " From whence the square of A E will be double * to the * Per 47. l. 1. " Square of A B, which is half of A C, and the square of " E Q double to the square of G Q or B F the intermediate " line. But the squares of A E and E Q are † equal † By the same. " to the square of A Q, that is, to the squares of A F and " F Q or F C the unequal parts. *Q. E. D.*]

[" Let the number 32 be divided equally into 16 and " 16, and unequally into 20 12. The square 20×20=400, " with the square 12×22=144, are double to the squares " of 16×16=256 and 4×4=16.]

PROP. X. Theorem.

IF a right line (F I) be divided into two equal parts in (L) and to it a certain right line (as I O) be adjoin'd; the square of the whole compound line (F O,) taken together with the square of the additional line (I O,) shall be double to the squares, which are described upon the half line (F L,) and (L O) that which is compounded of half the line (F I) and the additional line. Fig. 10.

[" For a construction being supposed not unlike to the Fig. 26. " former; the square of F E, is double to the square of " the half line F L, and the square of F G is double to " the * square of E Q or L O, which is compounded of * Per 47. l. 1. " the half line and the additional one. But the squares " of F E and E G are equal to the square F G; that is, " to the square of F O, the whole compound line, taken " together with the square of O G or O I the additional " line. *Q. E. D.*]

[" Let the number 40 be divided into 20 and 20, and " to it let there be added the number 14. The square " 54×54=2916, with the square 14×14=196 are dou- " ble to the square of 20×20=400, taken together with " 34×34=1156.]

PROP. XI. Theorem.

Fig. 11. *SO to cut the given right line (A B) in (C) that the rectangle (A B C) which is contain'd under the whole line and one part, ſhall be equal to the ſquare of the other part (A C.)*

From A erect a perpendicular A F equal to A B. Biſect A F in X. Draw the right line X B; from the line F A draw forth, cut off X I equal to X B. Then cut off A C equal to A I. I ſay the thing is done.

For let the ſquare B A F S be perfected; and a perpendicular being drawn through C, let the rectangle F I L O be perfected alſo. Becauſe F A is biſected in X, and to it is added A I; there ſhall be

(a) *Per* 6. *l.* 2.

{The rect. F I A + Square of X A} = (*a*) to the ſquare of X I

(b) *By the Conſtruction.*

That is, = to the ſquare of X B (*b*)

That is, = to the ſquares of {A B, A X} (*c*)

(c) *Per* 47. *l.* 1.

Therefore let there be taken away on both ſides the ſquare of X A; there will remain

the rectangle F I A or F L.

= A S the ſquare of the line B A;

Wherefore again, the common rectangle A O being taken away,

A L will remain equal to C S.

But A L is the ſquare of the line A C, ſeeing by the conſtruction A C and A I are equal. And C S is the rectangle A B C, foraſmuch as B S is equal to A B. Therefore the rectangle A B C is equal to the ſquare of A C. Therefore we have cut the line A B, as it was required.

Scholium.

THE ten firſt propoſitions of this book are true alſo in numbers: But this eleventh cannot be exemplify'd in numbers; for no number can be ſo divided that the product of the whole multiplied by one part ſhall be equal to the ſquare of the other. The force of this ſection of a line is wonderful. For which, ſee *Prop.* 30. *Lib.* 6.

PROP.

PROP. XII. Theorem.

IN an obtuse-angled triangle (ACB,) the square of the side (AB) opposite to the obtuse angle (C,) exceeds the squares of the other sides (AC, CB), by the rectangle (BCF) twice taken; which same rectangle is comprized under (BC,) one of the sides containing the obtuse angle, and the line (CF) which is intercepted betwixt the perpendicular (AF) and the obtuse angle. Fig. 12.

The square AB is equal to the squares of AF, BF (*a*.) (a) *Per* 47. *l.* 1.

But the square of BF is equal to the squares of FC, CB, with the rectangle FCB twice taken (*b*.) Therefore if you substitute these for the square of BF; then the square of AB is equal to AF square, FC square, CB square and rectangle BCF twice. (b) *Per* 4. *l.* 2.

But the squares of AF, FC are (*c*) equal to the square of AC. Wherefore this being substituted for them, (c) *Per* 47. *l.* 1.

AB square is equal to AC square, CB square + rectangle BCF twice.

PROP. XIII. Theorem.

IN any triangle whatsoever (as ACB) the square of the side (AB) opposite to an acute angle (C) is exceeded by the squares of the other sides (AC, CB) by the rectangle (BCF) twice taken; which same rectangle is contain'd under (BC) one of the sides comprehending the acute angle (C;) and the line (FC) which is intercepted betwixt the perpendicular (AF) Fig. 13, 14.

let

let fall upon the ſide (BC) from its oppoſite angle (A) and the acute angle (C.)

(d) *Per* 4. *l.* 2. The ſquare of BC is equal to (*d*) the rect. BFC twice + FC ſquare + FB ſquare

(e) *Per* 47. *l.* 1. And AC ſquare is equal to (*e*) CF ſquare + FA ſquare

Wherefore the two together {BC ſqu. AC ſqu.} are equal to rect. BFC twice + BF ſquare + 2 FC ſquare + AF ſquare

(a) *Per* 3. *l.* 2. But the rectangle BFC twice, together with the ſquare of FC twice, is (*a*) equal to the rectangle BCF twice. Therefore this being ſubſtituted for them.

BC ſqu. + AC ſqu. are equal to the rectang. BCF twice + BF ſquare + AF ſquare

(b) *Per* 47. *l.* 1. But the ſquares of AF, BF are equal to (b) the ſquare of AB. Therefore this being ſubſtituted for them.

BC ſqu. + AC ſqu. are equal to the rectangle BCF twice + AB ſquare

That is, BC ſquare + AC ſquare do exceed AB ſquare by the rectangle BCF twice taken.

Corollary.

Fig. 15. THE propoſion is true, although the perpendicular falleth without the triangle. And the demonſtra-

(c) *Per* 12. *l.* 2. tion is almoſt the ſame.

[" More briefly thus. ACq = (*c*) ABq + CBQ +

(d) *Per* 3. *l.* 2. " = ABq + 2CBq + 2CBF = (*d*) ABq + 2BCF.

" *Q. E. D.*]

Scholium.

Scholium.

FROM [illegible] proposition, and the 47th of the former Book, [illegible] have the measure of any triangle whatsoever, whose three sides are known, although the area be altogether inaccessible. For by the help of these theorems, the perpendicular is known, albeit the impediments of the place should not permit us to mark it out. But note, That the perpendicular, multiplied by half the side on which it falls produceth the area of the triangle, as appears out of the *Scholium* of the 41st proposition, Lib. 1.

Let there be any triangle (as A B C) having its sides known. It is required to give the perpendicular A F, which falls from the given angle A upon the opposite side C B. *Fig.* 15. *or* 14.

Take the square of the side A B opposite to the acute angle C, out of the sum of the squares of A C, and B C. By the 13th, the remainder shall be the rectangle B C F twice taken. Divide half of the remainder, that is, the rectangle B C F by the known side B C; thence will arise the right line C F. Take the square of the right line C F out of the square of A C. The remainder will give (*a*) the square of A F, whose square root will give the perpendicular A F. (a) *Per Prob.* 2. *Schol. post.* 47. *lib.* 1.

This thing also may be obtained out of the 12th proposition. [illegible] the 13th sufficeth, forasmuch as in every triangle the perpendicular let fall from some one of the angles into the opposite side, falls within the triangles.

PROP. XIV. Problem.

THE right-lined figure (Q X Z) being given, to make a square equal to it. *Fig.* 68.

Make (*b*) a rectangular parallelogram C I equal to the rectilinear Q X Z; the sides of which parallelogram, if they shall be equal, you have already made the square which was required; if they be unequal draw forth the greater side I A unto L, until A L shall be equal to A C. Then bisect I L in Z; from which, as from a center through (b) *Per* 45. *l.* 1.

through I and L describe a circle, and let C A be produced till it meets the circumference in B. The square of the right line A B is equal to the given rectangle Q X Z.

(c) *Per* 5. *l.* 2. For let the right line Z B be drawn; because I L is cut equally in Z, and unequally in A; the rectangle.

(d) *By the Construction.* I A L + Z A square } are equal (*c*) to Z L square, that is, equal to (*d*) Z B square, that is, equal to (*e*) Z A square + A B square.

(e) *Per* 47. *l.* 1. Taking away therefore on both sides the common Z A *q.* there remains.

Rect. I A L equal to A B *q*; that is,

Because A C and A L are equal, the rect. C I equal to A B square, and consequently A B square equal the rectilinear (*g*) Q X Z.

(g) *By the Construction.*

Scholium.

EUCLID's construction of this problem requires that the given rectilinear reduced unto a rectangle by *Prop.* 45. *l.* 1. which reduction being operose enough, the problem perhaps will more readily be dispatch'd after this manner. Let the given rectilinear be resolv'd into as many quadrangles (X, Z) as it can. Then to each quadrangle (*a*) make an equal rectangle. If there remain, as it here happens, one triangle (Q.) to it also (*b*) make a rectangle equal. Then to each rectangle by this 14th, *l.* 2. make an equal square; and lastly, to all these squares let one equal one be made (*c*). This will be equal to the given rectilinear Q X Z.

(a) *Per Schol. P.* 45 *l.* 1.

(b) *Per Corol. p.* 42. *l.* 1.

(c) *Per Prob.* 1. *Schol. p.* 47. *l.* 1.

THE

THE ELEMENTS OF EUCLID.

BOOK III.

THE fundamental properties of the moſt perfect amongſt plain figures are demonſtrated in this Book. The uſefulneſs of the book is manifeſt by this one thing alone, that it treats of a circle, that abundant ſource of admirable things through the whole mathematics. The more famous theorems are 16, 20, 21, 22, 31, 32, 35, 36.

DEFINITIONS.

1. THOSE circles are equal, whoſe diameters or ſemi-diameters are equal. *Fig. 20 l. 3.*

2. A right line (F B) is ſaid to touch a circle, when it doth ſo meet it in the point (B) that albeit it be produced, it doth not cut it.

3. Circles are ſaid to touch one another, when they do ſo meet that they do not cut each other. *Fig. 13, 14.*

4. In a circle the right lines (B C, F L) are ſaid to be equi-diſtant from the center (A,) when the perpendiculars which are let fall upon them from the center (A O, A I) are equal. *Fig. 18.*

5. Segments or portions of a circle are the parts into which the right line (C E) which cuts the circle doth divide it. *Fig. 37.*

6. An angle in a ſegment is that (B Q C) which is contain'd under the right lines, which are drawn unto one point of the circumference (Q) from the ends of the ſegment (B C) *Fig. 33.*

7. The

Fig. 33. 7. The angle (CQB) is ſaid to ſtand upon the circumference (BOC,) as being oppoſite to it.

Fig. 11. 8. A ſector is that part of a circle which is contain'd by two ſemi-diameters, as (AB, AF) and an arc (BF or BCF) intercepted betwixt the ſemi-diameters.

PROPOSITION I. Problem.

Fig. 1. l. 3. *To find the center of a given circle.*

Let the right line (BC) be drawn in the circle at random, which biſect in Q. Through Q draw the perpendicular LF, which biſect in A. A ſhall be the center.

If you deny it; let the center be O, which is without the right line FL (for in FL it cannot be, foraſmuch as this line is divided every where unequally but in A) and let there be drawn BO, QO, CO. Becauſe therefore you ſuppoſe O to be the center, BO, CO muſt be equal; and the triangles BOQ, COQ muſt be equilateral to each other; ſeeing by the conſtruction BQ and CQ are equal, and QO is common. Therefore the angle OQC (*a*) is equal to the angle OQB. Therefore OQC is a right angle (*b*) and conſequently equal to LQC, which is a right one by conſtruction, a part to the whole. Which is abſurd.

(a) *Per* 8. *l.* 1.
(b) *Per Def.* 14. *l.* 1.

Corollary.

FROM what hath been demonſtrated it appears, that if the right line (LF) cuts another right line BC into two equal parts and perpendicularly, the center is in the line that cuts the other.

Fig. 2. The center of a circle is very eaſily found by a ſquare; the top of it (Q) being applied to the circumference; for if the right line DE, joining the points D and E, in which the ſides of the ſquare cut the circumference, be biſected in A, (A) ſhall be the center. The demonſtration whereof depends on the 31ſt propoſition, *lib.* 3.

PRO-

PROP. II. Theorem.

IF in the circumference of a circle there be taken two points (C and B) the right line which is drawn through them falls entirely within the circle.

Let there be taken in the line B C any point whatſoever, *Fig.* 2.
as O, and from the center A, be drawn AB, AO, AC. Becauſe AB, AC are equal, the angles alſo B and C are (*c*) equal. Becauſe therefore A O C is (*d*) greater than the (c) *Per* 5. *l.* 1.
internal one B, it ſhall be greater alſo than C. In the triangle therefore OAC, the ſide AC ſubtending the greater (d) *Per Corol.* 1. *Prop.* 32. *l.* 1.
angle AOC, is (*e*) greater than the ſide AO, ſubtending the leſſer angle C, Seeing therefore AC reaches no farther than from the center to the circumference, AO ſhall not reach ſo far. Therefore the point O ſhall fall (e) *Per* 19. *l.* 1.
within the circle. The ſame thing may be ſhewed of any other point of the line BC. Therefore BC falls wholly within the circle.

The propoſition is alſo manifeſt from the very notion of a right line and a circle.

" Corol. " Hence it follows, that a right line touching " a circle, toucheth in one ſingle point only. For if it " toucheth the circumference in two points, it would be a " right line drawn thro' two points of the circle, and " conſequently would fall within the circle, contrary to " the definition of a tangent. And by the like reaſoning " (in paſſing from planes to ſolids) it might be prov'd, that " every plane toucheth a ſphere only in one point.

PROP. III. Theorem.

IF in a circle a right line (B L) drawn thro' the center biſects another (C F) not drawn through the center, it will cut it perpendicularly, and if it cut it perpendicularly, it will biſect it. Fig. 3.

Part I. From the center (A) let there be drawn AC, AF. The triangles X and Z are equilateral to each other,

other. For CO, FO are by the hypotheſis equal, and AC, AF are ſo, becauſe drawn from the center; while AO is common to both. Therefore the angles AOC,
(a) *Per* 8. *l.* 1. AOF are (*a*) equal. Therefore right (*b*) angles. Which was the firſt part.

(b) *Per Def.* 14. *l.* 1. Part II. Becauſe by the hypotheſis AOC, AOF are equal angles; AC ſquare ſhall (*c*) be equal to the ſquares of AO, OC together; and AF ſquare equal to the ſquares
(c) *Per* 47. *l.* 1. of AO, FO together. Seeing therefore the ſquares of AC, AF are equal, the ſquares of AO, OC together, will be alſo equal to the ſquares of AO, OF together: Wherefore taking away the common ſquare AO, the ſquares of OC, OF remain equal. And therefore the right lines OC, OF are equal. Which was the other part.

Corollary (1.) "Hence in every equilateral triangle, "and in that alſo which is only an *Iſoſceles*, a line which "falling from the top of the angle, biſects the baſe, is "perpendicular to it. And on the contrary, a line which "falling from the top of the angle is perpendicular to the "baſe, doth biſect it.

(2.) "Hence it follows, that half of the chord of every "arch, is the right ſine of half the arch.

PROP. IV. Theorem.

Fig. 4. 5. *IF in a circle two right lines, (BC, FL) not drawn both of them through the center, cut each other, they cannot biſect each the other.*

Fig. 5. For if one of them LF paſſeth through the center, it is manifeſt that it ſhall not be biſected by BC which doth not paſs through the center.

Fig. 4. If neither of them paſſes through the center, from the center A draw AO. If now BC, FL, were both biſected,
(a) *By the foregoing.* in O, the angles AOC, AOL would (*a*) be right angles, and conſequently equal; the whole to a part, which is abſurd.

PROP. V, VI. Theorems.

Fig. 6, 7. *CIrcles cutting each other, or inwardly touching one the other, have not the ſame center.*

For

For if it were otherwiſe, the right lines AB, ACF, drawn from the common center A, would be equal; and AC would be equal to AF; a part to the whole, becauſe they are both equal to AB. Which is abſurd.

PROP. VII. Theorem.

IF in a circle there be taken any point beſides the center (A,) as the point (C,) and divers right lines fall from thence unto the circumference (as CB, CL, CO, CF;) Fig. 8.

1. *(CB) which paſſeth through the center will be the greateſt.*

2. *The remaining part of the diameter (CF) will be the leaſt.*

3. *Of the reſt that will be the greater, which, is nearer to the greateſt.*

4. *And no more than two equal lines can be drawn from the ſaid point (C) which is different from the center, unto the circumference.*

Part I. Let AL be drawn from the center A. Becauſe AL, AB are equal, the common line AC being added to each. AC and AL together are equal to CB. But AL + AC are greater than LC (*b*.) Therefore CB is greater than LC. In the ſame manner BC will be ſhewed to be greater than any other. (b) *Per* 20. *l.* 1.

Part II. From the center A draw AO. AO (that is, AF) is leſs than AC, CO (*c*.) Therefore taking away the common line AC, CO remains greater than CF. In the ſame manner CF is prov'd to be leſs than CQ, or any other. (c) *By the ſame.*

Part III. In the triangles COA, CLA, the ſides LA, AC, are equal to AO, AC, each to each. But the angle LAC is greater than the angle OAC. Therefore (*d*) the baſe LC is greater than the baſe OC. (d) *Per* 24. *l.* 1.

Part IV. This is manifeſt from what goes before. For if there could be three drawn equal, CO, CI, CQ, there would be two on the ſame ſide equal: Which is contrary to part III.

Corollary.

Corollary. "By the foregoing reasoning *Theodosius* gathered, that of the arches of great circles drawn upon the surface of a sphere, from any point diverse from the pole of a certain circle unto that circle, the greatest is that which passeth thro' the pole of that circle; the least, that which is drawn unto the opposite point; and of the rest, that is the greater, which is nearest to the greatest; as also that no more than two equal arches can be drawn from that point unto the circle. And in the like manner may the reader reason of himself on some other of the propositions of this book; it being very easy to pass from planes to solids in these augmentations.

PROP. VIII. Theorem.

Fig. 9, 10. *IF from a point (A) taken without a circle there be drawn unto the circle the right lines (A B, A C, A F,) or (A O, A Q, A R;)*

1. *Of those which fall upon the concave circumference, the greatest is (A B) which passes through the center (Z.)*

2. *Of the rest that is the greater, which is nearer to the greatest (A B.)*

3. *Of those which fall without the circle, or upon the convex periphery, the least is (A O) which being produced would pass through the Center (Z).*

4. *Of the rest, that which is nearer to the least, is less than that which is farther off.*

5. *No more than two equal lines can be drawn unto the circumference from the same point (A,) whether they fall within the circle or only without.*

Fig. 9. Part I. From center Z draw Z C; because Z C, Z B are equal, the common A Z being added to each, A Z + Z C are equal to A B. But A Z + Z C are (a) greater than A C. Therefore A B is greater than A C. In like manner A B will be shewed to be greater than any other whatsoever.

(a) *Per* 20. *l.* 1.

Part II. Draw Z F. Because in the triangles A Z C, A Z F, the sides A Z, Z C are equal to A Z, Z F, ea-

to each, but the angle A Z C is greater than A Z F, therefore the base A C (*b*) will be greater than the base A F. (b) *Per* 24. *l.* 1.

Part III. Draw Z Q. The two lines A Q, Q Z are greater than A Z (*c*.) Taking away therefore the equals Z Q, Z O, there remains A Q greater than A O. In the same manner A O is proved less than any other. *Fig.* 10. (c) *Per* 20. *l.* 1.

Part IV. Draw Z R. The right lines A Q, Q Z are less than A R, R Z (*d*); therefore the equals Z Q, Z R being taken away, A R remains greater than A Q. (d) *Per* 21. *l.* 1.

Part V. This is manifest from the foregoing.

PROP. IX. Theorem.

IF from some point within a circle (as A) more than two equal right lines can be drawn unto the circumference; that point is the center. *Fig.* 11.

This is manifest from part IV. of the 7th proposition.

PROP. X. Theorem.

CIrcles cut each other in two points only. *Fig.* 12.

For let them cut, if it may be, in more (B, C, F) from A the center of the circle L Q, let there be drawn to the points B C F, the lines A B, A C, A F; these will be equal. Because therefore from the point A, within the circle O S, there are drawn three equal lines, A B, A C, A F, unto its circumference, A must also be the center (*a*) of the circle O S. Therefore the circles L Q, O S, which cut one another, have the same center. Which contradicts the 5th proposition. (a) *By the foregoing.*

PROP. XI. Theorem.

IF two circles touch each other inwardly, a right line drawn through their centers (A and I) passes through the point of contact (B.) *Fig.* 13.

If you deny it, let the centers have, if it may be, that situation, that a right line passing through them, shall fall

without the contact B, cutting the circles in O and L: Let the center be A and C; and join AB, CB. Because therefore CB, CO are equal, the common AC being added to each of them, AC + CB shall be equal to AO. But AC, CB are (*b*) greater than AB, that is than AL. (*c.*) Therefore also AO is greater than AL, a part than the whole. Which is absurd.

(b) *Per* 20. *l.* 1.
(c) *By the Definition of a Circle.*

PROP. XII. Theorem.

Fig. 14.

IF circles touch one another on the outside, a right line, which joins the centers, must pass through the point of contact.

If it be denied, let the centers be so placed, as for instance in A and B, that the line passing through them shall not pass through the contact S, but cut the circles in O and Q. Let the points A, S and B, S be joined. Then AS, BS together will (*d*) be greater than AB. But AS is (*e*) equal to AO, and BS equal to BQ. Therefore AO and BQ together will be greater than AB, a part than the whole. Which cannot be.

(d) *Per* 20 *l.* 1.
(e) *By the Definition of a Circle*

[Corollary. " A right line drawn from the center of one " of the circles through the point of contact, will pass " through the center of the other.]

PROP. XIII. Theorem.

Fig. 15, 16.

CIrcles touch both one another, and a right line in a point only.

Fig. 15.

For let two circles touch one another inwardly in a part of the circumference LC, if it may be: Then a right line drawn through the centers A and B will (*f*) pass through the point of contact, as in C. Let there be drawn also AL, BL. Because therefore BL, BC are equal (for they are drawn from the center B unto the circumference OLC), the common line AB being added, AB, BL shall be equal to AC. But AC is equal to AL, for they are both drawn from the center A unto the circumference LQC. Therefore AB, BL are equal to AL, contrary to *Proposition* 20. *L.* 1.

(f) *Per* 11. *l.* 3.

Then

Then let the two circles touch one another on the outside, in the arch OL, if it may be. The right line AB, joining the centers, will pass through the point of contact (a) as in O, for instance; Let AL, PL, be drawn. The two sides of the triangle AL, PL, will be equal to AO, PO, or the whole AP; contrary to *Proposition* 20. *L*. 1. Fig. 16. (a)Per 12. l. 1.

Lastly, let the right line BF, and the circle touch each other, if it may be, in some part (CE.) Let there be drawn unto the center the right lines CA, EA. The lines CA, EA will then be equal: And therefore the triangle CAE is an *Isosceles*. Wherefore the angles C and E (*b*) are acute. And therefore a perpendicular let fall unto BF from the center A, will fall betwixt E and C, (*c*) as, for instance, in D. There will therefore both AC and AE be equal to the perpendicular AD: which is absurd, and contrary to *Corollary* 14. *p*. 32. and to *Proposition* 47. *L*. 1. (b)Per Cor. 11. Prop. 32. l. 1. (c) Per Corol. 3. Prop. 32. l. 1.

Corollary.

CIRCLES, whose centers are in the same right line, and which cut it in the same point B, do touch one another in that point only. Fig. 17.

This proposition is manifest from the very notion of the lines which are compared together. For neither can a right line and the curve circumference of a circle, or the divers curvatures of unequal circumferences, or two curvatures both convex, agree as to any part of themselves. But they would agree if they touched one another in some entire and proper part.

PROP. XIV. Theorem.

IN a circle, equal right lines (BC, FL) are equally distant from the center (A.) And what lines are equi-distant from the center are equal. Fig. 18.

From the center (A) let there be drawn (AC, AF.) Likewise (AO, AI) at right angles to BC, FL. Thus BC, FL shall be bisected (*d*) in O and I. (d)Per 3. l. 3.

Seeing therefore the whole lines BC, FL are supposed equal, the halves also OC, IF must be equal, and consequently the squares of them are also equal. Seeing therefore

therefore the ſquares of A C, A F are equal, and the ſquare of A C is equal to O C q, and O A q, as alſo the ſquare of A F is equal to I F q, and I A q (*a*): It follows, that the two ſquares O C q, O A q are equal to the two ſquares I F q, I A q. Wherefore taking away the ſquares of O C, I F (which before were ſhewed to be equal) the ſquare of A O remains equal to the ſquare of A I. Therefore the perpendiculars O A, A I are equal. Therefore (*b*) B C, F L are equi-diſtant from the center. Which was the firſt part. Then for the converſe of it:

(a) *Per* 47. *l.* 1.
(b) *Per Def.* 4. *l.* 3.

If the diſtances A O, A I are ſuppoſed equal, then the ſquares of the equal right lines being taken away, by the ſame ratiocination it will be ſhewed, that the remaining ſquares O C q I F q are equal, and conſequently that the right lines O C, I F are equal, which ſeeing they are * halves of the right lines B C, F L, theſe alſo muſt be equal. Which was the ſecond part.

* *Per* 3. *l.* 3.

PROP. XV. Theorem.

Fig. 19.

OF right lines deſcribed in a circle, the greateſt is the diameter; and of the reſt, that is the greateſt, which is the neareſt to the center.

Let there be any line, as R S different from the diameter F L. From the center draw A R, A S. The two, A R, A S, which are equal to the diameter, are (*c*) greater than R S. Therefore, &c.

(c) *Per* 20. *l.* 1.

Then let B I be nearer to the center than X Z. From the center unto them draw the perpendiculars A C, A Q. A Q ſhall be greater (*d*) than A C. Take therefore A O equal to AC, and through O draw RS perpendicular to AO, which (*e*) will be equal to BI; and let A R, A S, A X, A Z be join'd. Becauſe therefore A is the center, the ſides A R, A S ſhall be equal to A X, A Z. But the angle R A S is greater than the angle X A Z. Therefore the baſe R S, that is B I is greater than the baſe X Z (*f*.) *Q. E. D.*

(d) *Per Def.* 4. *l.* 3.
(e) *By the foregoing.*
(f) *Per* 24. *l.* 1.

PROP. XVI. Theorem.

Fig. 20.

A Right line (I F) which being drawn through the point (B) the extremity of the diameter (C B) is perpendicular thereto, falleth all of it without

without the circle, and toucheth it in (B). Neither can any right line be drawn betwixt itself and the circle unto the point of contact (B), but it shall cut the circle.

Part I. Let there be taken in the line I B F any point L, unto which, from the center A, draw the line A L. Because, in the triangle A B L, the angle A B L is a right one, by the hypothesis, A L B shall be acute (g). Therefore A L, which is opposite to the greater angle B, will be greater than A B, which is opposite to the lesser angle L (h). But A B reaching only to the circumference, therefore A L shall reach beyond the circumference; and consequently fall without the circle. Which was the first part.

(g) *Per Corol.* 5. *p.* 32. *l.* 1.
(h) *Per* 19. *l.* 1.

Part II. Below BF, if it may be, let R B fall wholly without the circle. Because F B A is a right angle by the hypothesis, R B A will be acute, and therefore A B is not perpendicular to B R. Therefore let there be drawn from the center A to BR, the perpendicular AO, which (*a*) will fall towards R, and cut the circle in Q. Therefore A B, which is opposite to the greater angle A O B, is greater than A O, which is opposite to the lesser, to wit, the acute angle O B A. But A B is equal to A Q: Therefore A Q also is greater than A O, a part than the whole.

(a) *Per Corol.* 3. *Prop.* 32. *l.* 1.

Corollary.

1. HENCE it appears again, that the contact of a right line and a circular one, is only in one point. *Fig.* 20.

2. If from centers taken in the same right line infinitely protracted, there be described through B infinite circles, as well lesser than the first B S C, as greater, they shall all touch the right line I F in the same one point B. *Fig.* 17.

3. Circles therefore growing into an amplitude greater than any given one, approach always, even unto infinity, nearer and nearer to the tangent, but are never join'd to it, otherwise than in one single point of contact; which thing, although it be most evident, is yet truly admirable.

Fig. 17. 4. From these things it is manifest, that every geometrical line whatsoever is infinitely divisible. For let there be drawn from some point of the diameter unto the tangent the right line A Q. Infinite circles having centers in the right line B A, infinitely produced, touch the right line I F by *Corollary* 2. of this, and one another by *Corollary*, p. 13. in one and the same point B, and consequently are no where joined, either amongst themselves, or with the right line I F, but in the point B only. Therefore it is necessary that they divide the right line A Q into infinite parts, that is into parts exceeding any number assignable.

Fig. 20. 5. The angle of contingence or contact L B Q (that, to wit, which is contained under the tangent and the circumference) cannot be divided by any right line.

Fig. 17. 6. Nevertheless, by circumferences touching the line I F in the same point, it may be divided and diminished infinitely. And in this, and the third *Corollary*, lies hid the whole mystery of asymptotes, that is, of a right line approaching unto an hyperbola, together with itself infinitely produced, unto a distance less than any given one, yet never concurring with it.

PROP. XVII. Problem.

Fig. 26. *FROM the given point (B,) to draw a right line, which shall touch a given circle (OQ.)*

From A the center of the given circle, let there be drawn into the point B, the right line A B, cutting the periphery in O. From the center A describe through B another circle B C, and from O draw O C perpendicular to A B, which may meet the other circle in C. Draw C A meeting the circle O Q in I. The right line drawn from B unto I, will touch the circle O Q.

For because the sides B A, I A, are equal to the sides C A, O A, and the angle A contain'd betwixt the equal sides is common to both. In the triangles I A B, O A C, the angles A O C, A I B are also (*a*) equal. Therefore A I B is a right angle. For A O C is a right one by the construction. Therefore BI (*b*) toucheth the circle in I.

(a) Per 4. 1.
(b) Per 16. l. 3.

Scholium.

Scholium.

BY the 31ſt following, from the given point O, a line touching a given circle (BQ) may be well drawn thus;

Let the right line, joining the given point O, and the center A, be biſected in P. Then from the center P, through A and O, deſcribe a circle, meeting the given one in B. The right line O B will touch the circle. *Fig.* 27.

For A B being join'd, the angle A B O in the ſemicircle is a right one by *Prop.* 31. Therefore by *Prop.* 16. O B toucheth the circle B Q.

PROP. XVIII. Theorem.

IF a right line (C L) touch a circle, a right line (AB) drawn from the center (A) unto the point of contact (B) is perpendicular to the tangent. *Fig.* 28.

If it be denied, let ſome other right line (as A F) be the perpendicular from the center A. This will cut the circle in O. Becauſe therefore the angle A F B is ſuppoſed to be a right one, ABF (*c*) muſt be acute. Therefore AB (that is, A O) is greater than A F (*d*); a part than the whole; which is abſurd.

(c) *Per. Corol.* 5. *p* 32. *l* 1.
(d) *Per* 19. *l.* 1.

PROP. XIX. Theorem.

IF a line (B C) touch the circle, and from the point of contact (A,) there be rais'd (A I) perpendicular to the tangent, the center will be in that perpendicular. *Fig.* 29.

If you deny it, let the center be without A I in Z; and from it let there be drawn unto the contact the line Z A. The angle Z A C will be a right one (*e*) and therefore equal to the angle I A C, which by the hypotheſis, is a right one; that is, the part will be equal to the whole; which is abſurd.

(e) *By the foregoing.*

PROP. XX. Theorem.

Fig. 30, 31, 32. *THE angle at the center (BAC) is double to the angle (BFC) which is at the circumference, when the ſame arch (BC) is the baſe of the angles.*

Fig. 30. Here are three caſes. In the firſt caſe, the ſides B A, B F coincide. And then becauſe A F, A C drawn from the
(a) *Per* 5. center are equal, there will be in the triangle Z, the
l. 1. angles F and C equal (*a.*) But B A C is equal to the two
(b) *Per* 32. angles F and C (*b.*) Therefore B A C is double of F.
l. 1. In the ſecond caſe, B A, C A fall within B F, C F, and
Fig. 31. then F A X being drawn, X A B, by the firſt caſe, is double of X F B; and X A C double of X F C. Therefore the whole B A C is double of the whole B F C.

Fig. 32. In the third caſe, B F cuts A C, and the angle B A C is without the triangle B F C. Here let F A L be drawn; By the firſt caſe, the whole L A C is double of the whole L F C, and L A B taken away, is double of L F B taken away. Therefore the remaining angle, B A C, is alſo double of the remaining one, B F C. *Q. E. D.*

Corollary. "Hence we gather, that the ſides of every
Fig. 53. "triangle are to each other as the ſines of the angles oppoſite to thoſe ſides reſpectively. Let E F G be any "triangle; about which let a circle be underſtood to be "circumſcrib'd (*c*) and from the center of the circle, let
(c) *Per* 5. "there be let down the perpendiculars A B, A C, A D,
l. 4. "which will * biſect the ſubtenſes. Now as EF is to EG,
* *Per* 3. *l.* 3. "ſo half E F (that is, E B) to half E G (that is, E D.)
† *Per Corol.* "But E B is the ſine of the angle † B A E, that is, of
2. *p.* 3. *l.* 3. "half the angle E A F, that is, of the whole angle
‡ *Per* 20. "E G F, ‡ oppoſite to the ſides E F; and E D is the ſine
l. 3. "of the angle E A D, that is, of half the angle E A G, "that is, of the whole angle E F G, which is oppoſite "to the ſide E G. Therefore E F is to E G, as the ſine "of the angle E G F, is to the ſine of the angle E F G. "*Q. E. D.* And from this one propoſition a great part of "trigonometry is deduced. Which thing will be worth "our obſervation.

Corollary

Corollary (2.) "From the former corollary we learn to measure the distance of the moon. For astronomical observations giving us the angle of the diurnal parallax * B C A, we find out the distance of the moon by the following proportion. As the sine of the angle A C B, is to the sine of the angle A B C; so is the semidiameter of the earth, B A, unto the moon's distance, A C. *Q. E. I.* — Fig. 86. *l.* 1. — *Cor. 16. p. 32. l. 1.*

Corollary (3.) "From the second corollary we learn also to measure the distance of the sun, for there being given by astronomical observations the angle of the menstrual parallax, (namely, that which is made when the moon appears precisely bisected) or the angle Z E O, and, together with this angle, the moon's distance, Z O, We find the distance of the sun by this analogy. As the sine of the angle Z E O, is to the sine of the angle E O Z; which sine is the radius: So is Z O, the moon's distance, unto Z E, the distance of the sun. *Q. E. I.* — *Fig.* 54.

PROP. XXI. Theorem.

THE angles (B Q C, B F C) which in a circle stand upon the same arch (B O C,) or which are in the same segment (B Q S C) are all equal among themselves. — *Fig.* 33.

Let first the segment B Q S C be greater than a semicircle. From the center A draw A B, A C. By the foregoing, the angle B A C, at the center, is double of each, B Q C, B F C. Therefore they all, B Q C, B F C, are equal (*a*). *Q. E. D.* — (a) *Per Axiom* 6.

Then let the segment B Q C be equal to, or less than a semicircle. In the triangles B Q I, C F I, because the angles vertically opposite at I are equal (*b*,) the sum of the rest, Q and R will be equal to the sum of the (*c*) rest, F and O. Wherefore, if from these equal sums there be taken away the angles R and O, which by the first part, are equal, as standing upon the same arch Q F, the angles which remain, Q F, must be equal. *Q. E. D.* — *Fig.* 34. — b) *Per* 15. *l.* 1. — (c) *Per Cor.* 10. *p.* 32. *l.* 1.

Corollary. "Hence we gather in opticks, that any line B C to the eye placed where you will in the cir-

" cumference

"cumference of the circle, whereof the line is a chord, "appears of the same magnitude; to wit, because it appears every where under an equal angle B Q C.

Scholium. "If of two equal angles, standing upon "the same arch, one of them be at the circumference, "the other also will be at the circumference.

Fig. 33, 34. "If it be denied, B Q C shall either be equal to the "angle B I C, on this side the circumference Q F, or to "the angle B E C, which is beyond the said circumference. "But the angle B I C, is (d) greater, and the angle B E C "(d) is less than the angle B Q C. Therefore, &c.

(d) Per Corol. 1. p. 32. l. 1.

PROP. XXII. Theorem

Fig. 35. *IN any quadrilateral inscribed in a circle (A B C F) the opposite angles make two right ones.*

Let B F, C A be drawn. The angle A B C, with the (a) two, O and X, make two right angles. But O is equal to I (b,) because it stands upon the same arch, B C: And again, X is equal to Z, because it stands upon the same arch, A B. Therefore A B C taken together, with the two angles I and Z, that is, with the whole opposite angle A F C, makes two right angles. *Q. E. D.*

(a) Per 32. l. 1.
(b) Per 21. l. 3.

Corollary (1.) "Hence, if one side of a quadrilateral, "described in a circle, be protracted, the external angle "will be equal to the opposite angle of the quadrilateral; "for the internal, added to either of them, makes two "right angles.

(2.) "Likewise a circle cannot be described about a "rhombus, because its opposite angles either fall short of, "or exceed two right angles.

(3.) "Likewise, if in any quadrilateral A B C F, the "opposite angles F and B are equal to two right ones, a "circle may be described about it. For (a) a circle will "pass through any three angles, C, F, A, and this, so that ¶ a fourth be equal to B; which cannot be, unless it doth "indeed pass through the point B †. Therefore it doth "pass through it.

(a) Per. 5. l. 4.
¶ Per 22. l. 3.
† Per Schol. p. 21. l. 3.

PROP.

PROP. XXIII, XXIV. Theorems.

ARE not neceſſary; and they treat of ſimilar ſegments, which cannot rightly be defin'd without proportions.

PROP. XXV. Problem.

TO perfect a given arch (A B C.) Fig. 36.

Let there be ſubtended at random the two right lines A B, C B; which biſect in I and L. From I and L raiſe perpendiculars meeting one another in O. This ſhall be the center of that circle whereof A B C is a portion.

For (a) the center is both in the line I X, and in the line L Z. Therefore it is in their common point O. (a) *Per Corol. p.* 1. *l.* 3.

The practice. From the center B, taken in the arch, deſcribe a circle; and with the ſame interval, from other centers in the arch, deſcribe two other circles, each of which cuts the former twice. Two right lines drawn through the interſections, and croſſing each other in O, will give the center.

PROP. XXVI, XXVII. Theorems.

IN equal circles, equal right lines (C E, F I) ſubtend equal arches; and if the arches are equal, the ſubtenſes are alſo equal. Fig. 37.

Theſe two propoſitions are plainly axioms, and need no demonſtration.

Corollary (1.) " If in a circle A B C D the arch A B be
" equal to the arch D C; A D will be parallel to B C. Fig. 55.
" For A C being drawn, the angles A C B, C A D, as
" ſtanding on equal arches, will be equal. *Wherefore * *Per* 27. *l.* 1.
" A D is parallel to B C. *Q. E. D.*

(2.) " The right line E F, which is drawn from the Fig. 56.
" point A, the middle point of ſome arch, and toucheth
" the

" the circle, is parallel to the right line BC, which sub-
" tends that arch. For from the center D, draw unto
" the point of contact A the right line DA, and join DB,
" DC. The side DG is common, and DB is equal to
" DC, and the angle BDA equal to the angle CDA,
" the arches BA, CA being supposed to be equal. There-
* Per 4. l. 1. " fore the angles DGB, DGC are equal *, and conse-
" quently are right angles. But the internal angles GAE,
† Per 18. l. 3. " GAF are also right angles †. Therefore BC, EF are
" parallel §. *Q. E. D.*

§ Per 28. l. 1.

PROP. XXVIII, XXIX. Theorems.

Fig. 38. *If in equal circles, the angles, whether at the centers (BAC, FLI) or at the circumference (BOC, FSI) be equal; the arches also (BXC, FZI) on which they stand are equal; and if the arches be equal, the angles also are equal.*

These two propositions also are plainly axioms, and need no demonstration.

PROP. XXX. Problem.

Fig. 39. *To bisect a given arch (ABC.)*

Draw AC, which bisect in O. From O draw the perpendicular OB, meeting the arch in B. I say the thing is done.

For let AB, BC be join'd. The sides AO, BO are by the construction equal to CO, OB; and the angles at O
(a) Per 4. l. 1. are equal, as being right ones. Therefore the bases AB, CB are equal (*a.*) Therefore the arches also (*b*) AB,
(b) Per 26. l. 3. BC are equal.

The practice. From the centers A and C describe with an equal interval, arches cutting each other in the point F and I, the right line drawn through these points will bisect the arch ABC.

PROP.

PROP. XXXI. Theorem.

THE angle (BCF) in a semicircle, is a right one; that in a segment greater than a semicircle, is less than a right one; that in a segment less than a semicircle, is greater than a right one. Fig 40.

Part I. From the center A draw A C. Because A B and A C are equal, the angles O and B are equal (c.) (c) Per 5. l. 1.
For the same cause the angles I and F are equal. Therefore the angle B C F is equal to B and F together. Seeing (a) therefore the three together make two right angles, B C F, which is half of two right angles, is one right angle. (a) Per 32. l. 1.

Part II. Let the segment L O B C be greater than a semi-circle, and in it let there be the angle C O L, and let L B, the diameter of the circle, be drawn. The angle C O L is less than that B O L, which by part I. is a right one. Therefore, &c. Fig. 41.

Part III. Let the segment L O X be less than the semi-circle L O B, and X O L be the angle in it. This will be greater than B O L, which is a right one. Therefore, &c. Fig. 41.

Corollary (1.) "Hence we may make a proof of the instrument, called a square, whether it be exactly rect-angular or not. For in what circle soever the top of the square is laid upon C, or any point of the circum-ference whatsoever, if the sides of it do pass through the points of the diameter B and F, the angle is a right one; otherwise not. Fig. 40.

(2.) "If the sides of a square be held continually upon the points B and F, in the mean while that the angle is moved round, first on one side, then on the other, the top of the angle C will describe a circumference of a circle, whose diameter is the line B F.

(3.) "Hence we learn to raise a perpendicular at the end of a line. Let B C be the line, C the point given, from whence a perpendicular is to be raised. From any point whatsoever, A as the center, let a circle be described passing through the point C, and cutting B C in any point,

"as

" as B. If the diameter B F be drawn, it is manifest that " the line C F is the perpendicular required. *Q. E. F.*

Fig. 57. (4.) " Hence it is manifest, that circles touching one " another inwardly, do cut all lines, as A D proportion- " ably; or so, that A C, the subtense of the lesser, is to " A D, the subtense of the greater circle, as A E, the " diameter of the lesser, is to A B, the diameter of the " greater. For there being drawn the subtenses E C, B D, " the triangles E A C, D A B are equi-angled. For the " angle A is common, and A C E, A D B are right ones, " as being angles in a semi-circle; and therefore A E C,

(b) *Per* *Corol.* 9. *Prop.* 32. *l.* 1. " A B D (*b*) are equal. The triangles therefore are similar, " by the fourth proposition of the sixth book, and A E : " A B :: A C : A D. *Q. E. D.*

Fig. 40. (5.) " In a right-angled triangle B C F, if the hypo- " thenuse B F be bisected in A, the right line A C cuts " the triangle into two equicrural ones, A C B, A C F, " and so a circle-described from the center A, through B, " must pass through C, the top of the right angle.

PROP. XXXII. Theorem.

Fig. 42, 43. *IF a right line (C F) touch a circle, and another (AB) which is drawn from the point of contact (A) cut it; the angle made by the tangent and the cutting line, is equal to the angles which is made in the alternate or opposite segment.*

That is, the angle C A B will be equal to the angle L, which is made in the segment A L B; and the angle F A B will be equal to the angle O, which is made in the segment A O B. For,

Fig. 42. First, let the line A B pass through the center. Here by *Prop.* 18. C A B is a right angle: And by *Prop.* 31. L is also a right one. Therefore C A B and L are equal.

Fig. 43. Then let the line A B not pass through the center. Let the line A Q therefore be drawn through the center, and B Q be join'd. Because the angle in the semicircle

(a) *Per* 31. *l.* 3. (b) *Per* 32. *l.* 1. A B Q (*a*) is a right one, B Q A taken together with B A Q will make one right angle (*b*.) But C A Q is also by *Prop.* 18. of this book a right angle. Therefore BQA with BAQ are equal to C A Q. The common angle therefore B A Q, being

being taken away, there remains B Q A, which is equal to L [c] equal to C A B. Therefore L and C A B are equal: Which is the first part to be proved. [c] *Per* 21. *l.* 3.

Then F A B and C A B make two right angles [*d*.] and in the quadrilateral B O A L, the opposites L and O make two right angles [*e*.] Therefore the two F A B, C A B are equal to the two O and L. Therefore there being taken away on one side C A B, on the other L, which have already been shewed to be equal, there remains F A B equal to O. Which was the other part to be proved. [d] *Per* 13. *l.* 1. [e] *Per* 13. *l.* 3.

PROP. XXXIII. Problem.

UPON a given line (BC) to make a segment of a circle, in which the angle shall be equal to an angle given. Fig. 44.

First, let there be an acute angle given, A B F, from B draw B L perpendicular to A B; and at C, the extremity of the line B C, make B C I equal to C B L [by 23. *l.* 1.] whose side shall cut B L in I. From the center I describe a circle through B: This circle will also pass through C, forasmuch as because of the equality of the angles at B and C, the sides likewise C I, B I are [by 6. *l.* 1,] equal, and the segment B Q C shall contain an angle equal to the given one A B F.

For because A B is perpendicular to the diameter B L, A B will touch the circle which B C cuts [*a*]. Therefore the angle in the segment B Q C is equal [*b*] to the angle A B F. [a] *Per* 18. *l.* 3. [b] *By the foregoing.*

But if the angle given be obtuse, as R B C, do as before and C O B will be the segment required.

PROP. XXXIV. Problem.

FROM a given circle to take away a segment containing an angle equal to a given one. Fig. 45.

Unto the diameter of the circle F A, draw the perpendicular B A L. Then [*e*] let A C be drawn, which may make the angle B A C equal to that which is given. This line A C shall cut off the segment A Q C, containing an angle equal to the given one, as is manifest from Prop. 32. [e] *Per* 23. *l.* 1.

PROP.

PROP. XXXV. Theorem.

Fig. 46. 47, 48. *IF in a circle two right lines (CL, BF) cut one another, the rectangle (COL) under the segments of one is equal to the rectangle (BOF) under the segments of the other. For,*

If they interſect each other in A, the center of the circle, the thing is manifeſt.

Fig. 46. If one of them CL paſſeth through the center A, and biſects the other BF, which doth not paſs through the center; it (*f*) cuts it perpendicularly, and so the ſquare of FO is the ſame with the rectangle FOB. Let AF be drawn. Becauſe CL is biſected in A, and otherwiſe divided in O.

(f) *Per* 3. *l.* 3.

It will be thus,

(a) *Per* 5. *l.* 2.

Rect. COL + AOq. } will be equal to ALq (*a.*)

that is, AFq.

(b) *Per* 47. *l.* 1.

that is, equal to AOq. + FOq. } (*b.*)

Therefore the common ſquare AO being taken away, there will remain

Rect. COL equal to FOq, that is, to the Rect. FOB.

Then if one of the right lines CL paſſes through the center, and cuts the other BF unequally in O, let a right line drawn from the center A cut BF into two equal parts in I. In this caſe, AIB will be a right angle (*c.*) Now, becauſe CL is biſected in A, and otherwiſe in O, it will be thus:

Fig. 47.

(c) *Per* 3. *l.* 3.

Rect. COL + AOq. } will be equal to ALq. (*d*) that is, to ABq. that is, to AIq. + BIq. } (*e.*)

(d) *Per* 5. *l.* 2.

(e) *Per* 47. *l.* 1.

But AOq is equal to OIq + AIq. (*f.*) Therefore,

(f) *By the ſame.*

Rect. COL + OIq. + AIq. } equal to AIq. + BIq. }

Therefore the common ſquare AI being taken away, there remains,

Rect.

Rect. COL = BIq.
— OIq

But BI square is equal to the Rectangle FOB, together with OI square; (g) because FB is bisected in I, and otherwise cut in O. Therefore, (g) *Per.* 5. l. 2.

Rect. COL } are equal to Rect. FOB }
+ OIq. } + OIq. }

Therefore the common OIq. being taken away; there remains

Rect. COL. = Rect. FOB.

But lastly, if neither of the Lines CL, FB, passes through the Center; Through their common Intersection (O) let there be drawn the right Line XZ, which passes through the Center (A). By what hath been just now demonstrated, both the Rectangle COL, and that FOB, are equal to the Rectangle ZOX. Therefore COL, FOB, are equal betwixt themselves (h). *Fig.* 48. (h) *Per Axiom* 1.

[" Or the Proposition may be demonstrated more easily " and universally thus. Join AC and BD. Here be- " cause of the Equality of the Angles CEA, BED, as " being vertically opposite (a); and of the Angles C and " B, as being upon the same Arch AD †; The Triangles " CEA, BED are equi-angled (*per Corol.* 9. *p.* 32. *l.* 1.) " Therefore * CE : EA :: EB : ED. Therefore CE× " ED is equal to EA×EB (*per* 16. *l.* 6.) *Q. E. D.* *Fig.* 58. (a) *Per* 15. *l.* 1. † *Per* 21. *l.* 3. * *Per* 4. *l.* 6.

PROP. XXXVI. Theorem.

IF from (B) a Point given without a Circle, there be drawn unto the Circle two right Lines one (BF) touching it, the other (BC) cutting it; the Rectangle (CBO,) which is comprehended under the whole cutting Line (CB) and the Part (BO,) which lies betwixt the Point (B) and the Circle, is equal to the Square of the Tangent (BF) *Fig.* 49. 50 51.

1. If the cutting Line BC passes through the Center A, join AF. This, with the Line FB, will make a right Angle (a). And therefore because CO is bisected in A, and to it is added OB; it will be thus: (a) *Per* 18. *l.* 3.

Rect. CBO } will be equal to ABq (b) that is, (b) *Per* 6. *l.* 2.
+AOq }

to AFq }
+ FBq } (c.) (c) *Per* 47. *l.* 1.

Therefore the equal Squares A O q, A F q being taken away on both Sides, there remains,

Rect. C B O = B F q.

Fig. 50. 51. 2. But then, if C B doth not pass through the Center, let there be drawn A B, A F, A O and A L, and let A L bisect O C in L. The Angle A L O is therefore a right one (*d*). Likewise A F B is a right Angle (*e*). Now, because C O is bisected in L, and to it is added O B, it will be thus:

(d) *Per.* 3. *l.* 3.
(e) *Per.* 18 *l.* 3.
(f) *Per* 6. *l.* 2.

$$\left.\begin{array}{l}\text{Rect. } CBO\\ +LOq\end{array}\right\} = LBq\ (f.)$$

Let there be added on both Sides A L Square, and then

$$\left.\begin{array}{l}\text{Rect. } CBO\\ +LOq\\ +ALq.\end{array}\right\} \text{equal to } \left\{\begin{array}{l}LBq.\\ +ALq.\end{array}\right.$$

(g) *Per* 47. *l.* 1. But the Squares of L O, A L are equal (*g*) to the Square of A O, or A F; and the Squares of L B, A L are equal to the Square of A B (*h*). Therefore,

(h) *By the same.*
(i) *By the same.*

$$\left.\begin{array}{l}\text{Rect. } CBO\\ +AFq.\end{array}\right\} = ABq. \text{ that is, } (i).$$

$$\text{to } \left.\begin{array}{l}BFq.\\ +AFq.\end{array}\right\}$$

Therefore the common Square, that of A F being taken away, there remains,

Rect. C B O equal to the Square of B F. *Q. E. D.*

Fig. 59. [" Or more easily and universally thus. Draw A B
* *Per* 32. " and B C. Now because of the Equality of the Angles
l. 3. " A, and D B C, * and for that the Angle D is common to
† *Per Corol.* " both; the Triangles B D C, A D B are equi angled †.
9. *p* 32. *l.* 1. " And therefore (by *Lib.* 4. 6.) A D : D B :: B D : D C.
* *Per* 17. " Wherefore the Rectangle * A D×D C is equal to the
l. 6. " Rectangle B D×D B, or D B q. *Q. E. D.*]

Corollaries.

Fig. 52. 1. IF from the same Point B, without the Circle, as many cutting Lines B C as you will be drawn, all the Rectangles C B O are equal amongst themselves. For each of them is equal to the Square of the Tangent B F.

2. The right Lines B F, B Q, which, from the same Point, touch the Circle, are equal. For each of their Squares is equal to the same Rectangle.

3. " It

3. " It is also clear, that from the same Point B, taken " without the Circle, there can only two Lines B F, B Q " be drawn, which shall touch the Circle. For if a third " be said to touch it, it must be equal to B F, or B Q, " and therefore the same with one of them.

4. " In every Right angled Triangle B F A, that is not " also an Isosceles, the Rectangle arising from the Sum of " the Hypothenuse, and one Side drawn into the Diffe- " rence betwixt them, is equal to the Square of the other " Side. For the Sum of the Hypothenuse BA, + AF, or " AC, is = BC. And their Difference is BA—AF=BA " —AO=BO. And the other Side of the Triangle is " BF. But the Rectangle C B O is equal to the Square of " BF; Therefore, *&c.* *Fig.* 49.

PROP. XXXVII. Theorem.

IF the Rectangle under C B and O B be equal to the Square of B F, this must touch the Circle in F. *Fig.* 52.

From B let there be drawn the Tangent BQ, and the right Lines EQ, EF being drawn from the Center E, unto the Points Q and F, let BE be joined. Because by the Supposition the Square of B F is equal to the Rectangle CBO, as is also the Square of BQ, by 36th of this Book, the Squares of BQ, BF, shall be equal betwixt themselves and consequently the right Lines BQ, BF, are equal. Therefore the Triangles FEB, BEQ are Equilateral each to other. Therefore the Angles Q, F are equal (*a*). But Q is a right Angle, (*per* 18. *l.* 3.) Therefore F also is a right Angle. Therefore BF toucheth the Circle (*b*).

Corollaries (1.) " Hence the Angle E B F is equal to " the Angle E B Q (*per* 8. *l.* 1.)

(2.) " If two equal right Lines BF, BQ fall from some " Point B upon the convex Circumference, and B F, one " of them toucheth the Circle, the other BQ must touch " it also. For seeing B F, B Q are equal, their Squares " are also equal. But B F q is equal to C B O (*a*). There- " fore BQq=CBO (*b*). Therefore BQ also toucheth the " Circle (*c*).

(a) *Per* 8. *l.* 1.
(b) *Per* 1[illegible] *l.* 3.

(a) *By the foregoing.*
(b) *Per Axiom* 1.
(c) *By the Proposition*

Scholium [1.] " Seeing all Planes passing through the " Center of the Earth, which all stand perpendicular upon " the Horizon, do produce great and equal Circles upon " the Earth's Surface, we shall here bring in some elegant " Consectaries from thence, out of our Author, in his " Astronomy; which from the Nature of Circles may " very easily be understood.

(1.) " If in any Part the Surface of the Earth were per- " fectly plane, Men could no more stand upright upon it, " than upon the side of an Hill, saving in the Point of a " Contact only.

(2.) " The Head of a Traveller performs a longer Way " or Course than his Feet: Likewise he that is on Horse- " back, and goes the same Way as a Footman, measures " a greater or longer Space than he that is on Foot. As " likewise in a Ship, the uppermost Part of the Mast runs " over more Way than the lower Parts of it.

(3.) " If any one should travel over the whole Circum- " ference of the Earth the Way gone over by his Head " would exceed that which was gone over by his Feet, by " the Difference of Circumferences, or by the Circumfe- " rence of a Circle, whose Semi-diameter is the man's " own Stature.

(4.) " If a Vessel full of Water be elevated perpendicu- " larly, the Water will continually be running over, and " yet it will remain full; namely, because the Surface of " the Water is continually compressed into the Surface of " a greater Sphere. Yea, if a Vessel be elevated continually " higher and higher, the Surface of the Water which is " contained in it, will continually descend and come nearer " unto a Plane; unto which yet it will never actually come.

(5.) " If a Vessel full of Water be carried directly down- " wards, although nothing run over, yet it will cease to " be full; namely, because the Surface of the Water swells " continually into a Part of a lesser Sphere. From whence " it follows,

(6.) " That one and the same Vessel contains more Water " at the Foot of a Mountain than at the Top; as like- " wise more in a subterraneous Cellar, than in a Cham- " ber. To which Things add

(7.) " That two Strings, on which two Iron Balls hang " perpendicular, [and consequently the Walls of Buildings " erected perpendicularly] are not parallel one to another,

" but

" but Parts of Radius's meeting together in the Center of " the Earth.

Scholium [2.] " I think it not amiſs to inſert in this *Fig.* 60. " Place the following Problem alſo, which was communi- " cated to me by a Friend, as demonſtrated by me ſome- " what more briefly.

" Through the two Points (B) and (C) in a given Circle " (F D M) to draw the Circumference of a Circle which " ſhall biſect the Circumference of the other given Circle.

" Through the Center A, and one of the given Points " B, let there be drawn the infinite right Line B A M E. " Unto which, from the Center, let there be erected the " Perpendicular A D, and let B D be drawn. Let the " Line D E be made perpendicular to B D, cutting the " infinite Line B A M E in the Point E. Laſtly, let a Cir- (a) *Per* 5. " cle be drawn (*a*) through the three Points B C E. I ſay *l.* 4. " the Thing is done. For,

" Let the Chord of the ſecond Circle be drawn through " either of the Interſections of the Circles, as G, and " through A the Center of the firſt Circle, to wit, G A f; " let alſo the Diameter of the firſt Circle GAF be drawn, " Then in the firſt Circle (by *Corol.* 1. *Prop.* 8. *l.* 6. and " by *Prop.* 17 *l.* 6.) B A×A E=A D q, that is (becauſe " of the Equality of the Semi-diameters A D, A G, A F,) " =A G×AF. And in the ſecond Circle there will be " (*b*) AB×A E=A G × A f. Therefore A F= A f, and (b) *Per* 35. " the Points F and f will coincide, and the Arch F D G is *l.* 3. " equal to the Arch F M G. *Q. E. F.*

THE

Elements of EUCLID.

BOOK IV.

THIS Book, which is wholly Problematical, teacheth by what Artifice, Figures, thoſe which are ordinate or regular eſpecially, may be inſcribed in, and circumſcribed about Circles. There is very great Uſe of it in building Fortifications; and from it, as a Fountain have been derived thoſe moſt excellent Tables of Sines, Tangents and Secants, to the very great Benefit of the Mathematicks.

[" This Book is moſt uſeful for *Trigonometry*; for by in-
" ſcribing *Polygons* in a Circle, we learn to frame Tables of
" *Chords Tangents*, and *Secants*; by the help of which we
" learn to Meaſure the *Magnitudes* of *Figures* and *Bodies*.
" Neither without it can we duly *diſtinguiſh* the *Aſpects*, as
" they call them, of the *Stars* as the *Quartile*, *Sextile*, &c.
" they wholly depending upon the *Inſcription* of *Polygons* in
" *Circles*. Neither can we otherwiſe collect the *Area* (which
" is a certain *Quadrature* of a *Circle*) than from the *Area*'s
" or *Squares* of innumerable *Polygons* inſcrib'd in, and cir-
" cumſcrib'd about a *Circle*. And in like manner we know
" the *duplicate Proportion* of *Circles* amongſt themſelves, from
" the *duplicate Proportion* of *Polygons* inſcribed in, or cir-
" cumſcribed about *Circles*. And as for *Military Architec-*
" *ture*, it makes ſo much uſe of *Polygons* inſcrib'd in *Circles*,
" that more than all other Sciences it may ſeem to be whol-
" ly owing to this Book.]

DEFINITIONS.

1. A Rectilinear Figure is ſaid to be inſcrib'd in a Circle, or to have a Circle circumſcrib'd about it, when the Tops of all the Angles thereof are in the Circumference of the Circle.

2. A Rectilinear Figure is ſaid to be circumſcrib'd about a Circle, or to have a Circle inſcrib'd in it, when each one of its Sides toucheth the Circle.

3. An ordinate or regular Figure is that which is Equilateral and Equiangular.

PROPOSITION I. Problem.

TO inſcribe a right Line (A,) which is not greater than the Diameter, into a Circle (BD) Fig. 1. l. 4.

Take in the Circumference any Point B. From the Center B, with the Interval of the given Line A, deſcribe an Arch, cutting the Circle in C. Draw the right Line BC. I ſay the Thing is done.

PROP. II. Problem.

TO inſcribe in a Circle a Triangle having equal Angles with a given one (X). Fig. 2.

Let the Line EF touch the Circle in D. Let EDG be made (*a*) equal to the Angle C, and FDH equal to B; and join GH. I ſay the Thing is done. For (*b*) EDG is equal to H. H conſequently is equal to the Angle C (*c*). And FDH is equal (*d*) to G; and conſequently G to B. Therefore GDH (*e*) is equal to the Angle A. Therefore what was required is done.

(a) *Per* 23. *l.* 1.
(b) *Per* 32 *l.* 3.
(c) *By the Conſtruction.*
(d) *Per* 32 *l.* 3.
(e) *Per Corol.* 9. *p.* 32. *l.* 1

PROP. III. Problem.

TO circumſcribe about a Circle a Triangle, having equal Angles with a given one (IKL). Fig. 3.

Let the Line IK be drawn forth on both Sides, ſo as to make the external Angles O and N. Make at the Center A, the Angles GAB, BAF equal to O, N reſpectively, which

which is done by 23. *l.* 1. Then in the Points G, F, B, let three right Lines touch the Circle, meeting together in C, E, D. The Triangle C E D is circumscribed about the Circle, and is equi-angled to the given one I L K. For,

(f) Per 18 *l.* 3. In the Quadrilateral C G A B, the Angles G and B are (*f*) both of them right ones. Therefore the remaining ones G A B, and C taken together, do (*g*) make two right ones, and consequently are equal to the two together O, I. Therefore the two G A B and O, which are equal by the Construction, being taken away, there remains C equal to I. In the same manner E will be proved equal to the Angle K. Therefore D and L are (*h*) likewise equal. That therefore is done which was demanded.

(g) Per Theorem 1. Schol p. 32. *l.* 1.

(h) Per Corol. 9. Prop. 32. *l.* 1.

For that the Tangents do concur is thus shewed. The Angles O, I, and K, N are (*a*) equal to four right ones; and IK are less than two right ones (*b*). Therefore O, N, (that is, by the Construction G A B and B A F) are greater than two right ones. Therefore G A F (*c*) is less than two right ones. Therefore G F falls between A and D. Therefore seeing A G D, and A F D are right Angles, D G F, and D F G are less than two right ones. Therefore CGD and E F D † meet together towards D. In the same manner it may be demonstrated that the rest concur.

(a) Per 13 *l.* 1.

(b) Per 32 *l.* 1.

(c) Per Corol. 3. Prop. 13. *l.* 1.

† Per Schol. p. 3. *l.* 1.

Fig. 3.

PROP. IV. Problem.

TO inscribe a Circle in a Triangle.

Fig. 3. Bisect the two Angles C and E with the Lines C A, E A, meeting together in A. From A draw the Perpendiculars, A B, A G, A F. A Circle described from the Center A through B, will pass also through G and F, and touch the three sides of the Triangle.

For in the Triangles C A G, C A B, because the Angles A G C, ABC and likewise those G C A, and B C A are equal by the Construction and the Side A C is common, the Sides AG, AB * must be likewise equal. In like manner A B, A F may be shewn to be equal. Therefore the Circle describ'd from the Center A, passeth through B, G, F. And because the Angles at those three Points are equal, it toucheth (*f*) all the Sides of the Triangle. That therefore is done which was required.

* Per 26. *l.* 1.

(f) Per 16 *l.* 3.

["Hence

[" Hence the Sides of a Triangle being known, the Seg-
" ments of them, which are made from the Contacts of an
" inscribed Circle, will be known. Let D C be 12, D E
" 18, C E 16, D C and C E will be 28, from which sub-
" tract 18=D E=D G+BE, there remains 10=C G+
" C B. Therefore C G or C B=5. Consequently E B
" or E F=11. Wherefore F D or D G=7.]

PROP. V. Problem.

TO describe a Circle about a Triangle, or thro' three given Points, B, C, D, not lying in a right Line, to describe a Circle. Fig. 4.

Connect the given Points with two right Lines BC, CD, which bisect with the Perpendiculars EA, OA, meeting together in A. This will be the Center of a Circle which passes through B, C. D.

Let the right Lines A C, A D, A B, be drawn. By the Construction, the Sides DO, OA are equal to these, CO, AO; and the Angles at O are right ones. Therefore AD is equal to A C (*a*) In the same manner A B may be prov'd equal to AC. Therefore A D, A B, (*b*) are equal. Therefore a Circle described from the Center A through B will pass also through C and D. Which was the Thing required.

(a) *Per* 4 *l.* 1.
(b) *Per Axiom* 1.

As for the Practice, it is sufficient to describe from B, C, D, three equal Circles, intersecting each other; and through the Intersections to draw right Lines, these meeting one another will give the Center sought.

PROP. VI, VII. Problems.

TO inscribe a Square in, and circumscribe one about a Circle. Fig. 5.

Let the Diameters B D, C E, be drawn, cutting each other perpendicularly. The right Lines which join the Terms of these, inscribe a Square in a Circle.

The

The Demonſtration is manifeſt from 4. l. 1. and 31. l. 3. Then let four Tangents be drawn touching the Circle in B, C, D, E, meeting together in I, F, G, H. The Figure I F G H is a Square, circumſcrib'd about a Circle.

The Demonſtration is manifeſt from 18. l. 3. with *Corollary* 2. *Propoſition* 36. l. 3. and 28, and 34. l. 1.

Scholium.

Fig. 5. (c) Per 31. l. 3. (d) Per 47 l. 1.

A Square deſcribed about a Circle is double to that inſcribed. For becauſe the Angle B C D in the Semicircle (*c*) is a right one, the Square of B D (that is, F I Square) ſhall be (*d*) equal to B C q. + C D q, and therefore double to the Square of CD, *i. e.* to C D E B.

PROP. VIII. IX. Problems.

Fig. 6.

TO inſcribe a Circle in, and circumſcribe one about a Square, as (B C F E.)

Let there be drawn the Diameters of the Square, cutting each other in O. From the Center O deſcribe a Circle through B; this will alſo paſs through E, F, C.

Then from the Center O draw O D perpendicular to B C; a Circle deſcrib'd from the Center O through D, will touch all the Sides of the Square.

(c) Per 5 l. 1. (d) Per Corol. 11. pro. 32 l. 1. (e) Per 6. l. 1.

Part I. Becauſe by the Hypotheſis, the Lines C B, E B, are equal; the Angles BCE, BEC, will be equal (*c*). But C B E is a right Angle by the Hypotheſis B C E therefore and B E C are half right ones (*d*). In the ſame manner C B F will be ſhewed to be an half right Angle, as likewiſe the reſt of the Angles; and ſo they are equal amongſt themſelves. Therefore in the Triangle B A C, ſeeing there are two equal Angles C B O, B C O, the right Lines O B and O C (*e*) are equal. In the like manner the right Lines O B, O E, O F, may be ſhewed to be equal. Therefore a Circle deſcribed from the Center O through B, will paſs through E, F, C.

Part II. From O let there be alſo drawn the Perpendiculars O G, O H, O I. Becauſe in the Triangles GBO, DBO, the Angles at D and G, as likewiſe thoſe at B, are equal,

the

the Side O B is common, the Sides O D, O G, must be equal (*a*.) In the same manner O G, O H, O I may be shew'd to be equal. Therefore a Circle describ'd from the Center O, which passeth through D, will also pass through G, H, I, and touch all the Sides of the Square (*b*.) Because the Angles at D, G, H, I are right ones. Therefore we have done what was required.

(a) *Per* 26 *l.* 1.

(b) *Per* 16 *l.* 3.

PROP. X. Problem.

TO make an Isosceles Triangle BAC, in which the Angle at the Base (ABC, or ACB) shall be double to that which is at the Top (A). Fig. 7.

Let any right Line, what you will, as A B be taken, which so cut in D (*c*) that the Rectangle A B D shall be equal to A D Square. Then from the Center A, thro' B, describe a Circle; in which inscribe (*d*) B C equal to A D, and join A C. B A C shall be the Triangle sought.

(c) *Per* 11 *l.* 2.

(d) *Per* 1. *l.* 4.

For let the right Line D C be drawn, and through A, D, C, describe (*e*) a Circle. Because the Rectangle A B D is equal to the Square A D, (that is BC) it is manifest, that BC (*f*) toucheth the Circle DO, which C D cuts. Therefore the Angle B C D (*g*) is equal to the Angle A in the opposite Segment; and so the common Angle D C A being added, B C A must be equal to A+D C A. But because the Sides A B, A C are equal A B C (*h*) is equal to the Angle A C B. Therefore the Angle ABC is also equal to A+DCA. But the external Angle also BDC is equal to the two internal ones (*i*) A+DCA. Therefore A B C and B D C are equal. Therefore the Line D C is (*k*) equal to B C (that is by the Construction to D A.) Therefore the Angles A and D C A (*l*) are equal. Wherefore the Angle A B C which hath been shewed equal to those two, shall be double to one A. That is done therefore which was required.

(e) *Per* 5. *l.* 4.

(f) *Per* 37. *l.* 3.

(g) *Per* 32 *l.* 3.

(h) *Per* 5. *l.* 1.

(i) *Per* 32 *l.* 1.

(k) *Per* 6. *l.* 1.

(l) *Per* 5. *l.* 1.

Corollary.

Corollary.

EACH of the Angles at the Base B and C in the *Isosceles* now framed, is two Fifths of two right ones, or four Fifths of one right one, and the remaining one A is one Fifth of two right ones, or two Fifths of one right one. And is manifest out of this Proposition taken together with that 32. l. 1.

PROP. XI. Problem.

Fig. 7, 8. *TO inscribe a regular Pentagon in a Circle.*

(a) *By the foregoing.* (b) *Per* 2. *l.* 4. Let there be described (*a*) the Triangle BAC, having the Angles at the Base double to that at the Top. Inscribe a Triangle C A D equi angled to this in a Circle (*b*). Bisect the Angles at the Base ACD, ADC, with the right Lines CE DB, cutting the Circle in E and B. The Points A, B, C, D, E, join'd by right Lines, will give an ordinate Pentagon inscribed in a Circle

For from the Construction it appears that the Angles I, N, Q S, O, are equal. Wherefore the Arches subtended to them, A E, E D, C D, C B, B A, are also (*c*) equal. (c) *Per* 28 *l.* 3. Therefore the right Lines subtended to those Arches, shall also (*d*) be equal. The Pentagon therefore is equilateral. (d) *Per* 27. *l.* 3. But it is also (*e*) Equiangular, because its angles B A E, A E D, &c. stand on equal Arches B C D E, A B C D, (e) *Per* 29. *l.* 3. &c. that therefore is done which was required.

Corollary.

Fig. 8. THE Angle of a regular Pentagon makes six Fifths of one right Angle, or three Fifths of two. For the three Angles at A, seeing they are equal, as standing upon equal Arches, B C, C D, D E, and the middlemost of them by the Corollary foregoing is two Fifths of one right Angle; the three together, that is, the Angle of the Pentagon itself must make six Fifths of one right one.

" [Scholium. "This holds universally, that Figures of an *Fig.* 8.
" odd Number of Sides are inscribed in a Circle by means
" of an *Isosceles* Triangle, whose equal Angles at the Base
" are multiple of those at the Top. But Figures of an
" even Number of Sides are inscribed by the means of
" *Isosceles* Triangles, whose Angles at the Base are each
" of them multiple sesquialteral of that which is at the Top.
" As in the *Isosceles* A C D, if the Angle C or D be
" threefold of A the Side C D will be the Side of an
" Heptagon: if four-fold, it will be the Side of an Ennea-
" agon, &c. But if C or D shall be one and a half of A,
" CD will be the Side of a Square: and if C shall be two
" and a half of the Angle A CD will subtend a sixth Part
" of the Circumference: In like manner, if C or D shall
" be three and a half of the Angle A, CD shall be the side
" of an Octagon, &c.

Scholium.

EUCLID's Inscription of a Pentagon is ingenious, but that of *Ptolomy*, which he delivers in the first Book of his *Almagest*, is much more expeditious; and it is this: *Fig.* 12.

Let the Diameters ED, BF, be drawn, cutting one another perpendicularly in A. Bisect the Radius AD in C. From the Center C through B, describe an Arch, meeting the Diameter E D in G. The right Line G B is the Side of a Pentagon, and A G of a Decagon.

The Demonstration cannot be given here, for it depends upon the 13th Book of *Euclid*. See it in *Clavius*, in his Scholium, after *Prop.* 10. *l.* 13.

III. *Problem.*

UPON a given right Line (A B) to describe a regular Pentagon. *Fig.* 9.

Cut A B so in C (*a*) that the Rectangle A B C may be equal to the Square of A D. From A B protracted on both Sides take away A D, B E, equal to the greater Segment A C. From the Centers A and D with the Interval A B, describe two Arches, cutting each other in F. Likewise, from the Centers B and E, describe, with the same Intervals,

(a) *Per* 11 *l.* 2.

val, two Arches cutting each other in G And again, from the Centers G and F, with the same Interval, describe two others, cutting each other in I. The Points A, F, I, G, B, being joined, will give a regular Polygon upon the right Line AB.

That it is Equilateral, is manifest from the Construction; that it is equi-angled, will be thus demonstrated. Let DF be drawn. It is manifest by the Construction, that ADF is an *Isosceles*. And the Base AD is the greater Segment of the Side DF, so divided, that the Rectangle of the whole and the lesser Side, is equal to the Square of the greater. (For D F is equal to A B, and A D equal to A C.) Therefore the Angle DAF is two Fifths of two right ones; by *Corol. Prop.* 10. *l.* 4. Therefore the remaining Angle FAB is three Fifths of two right ones, or six Fifths of one right one (*b*); and therefore is an Angle of a regular Pentagon (*c*). In the same manner may it be shewn that the Angle GBA is three Fifths of two right ones, and so equal to F A B. From whence it is necessary, that the rest F, G, I, should be equal to these, as appears from their being Equilateral to these, if the right Line F G be conceived to be subtended, as appears by *Prop.* 8. *l.* 1.

(b) *Per* 13 *l.* 1.
(c) *Per Corol* 5. *p.* 11. *l.* 4.

PROP. XII. Problem.

Fig. 10. *TO circumscribe an ordinate Pentagon about a Circle.*

Let there, by the foregoing, be inscribed the regular Pentagon G H I K M, and let there be drawn Tangents in the Points G, H, I, K, M, which may concur in B, C, D, E, F, I say the Thing is done.

For from the Center draw the right Lines A G, A B, A H, A C, A I. Here because from the same Point B, BG, and BH touch the Circle, they (*a*) are equal. Therefore the Triangles G A B, B A H are Equilateral to each other. Therefore (*b*) the Angles O P, as likewise those Q, S, are equal And therefore the whole Angle B is double to P, and the whole G A H double to S. For the same reason the Angles C and H A I are double to T and N respectively. But G A H and H A I are equal (*c*), because they stand upon equal Arches; by Construction G H, H I. Therefore their halves, S and N are also equal

(a) *Per Corol.* 2. *p.* 36. *l.* 3.
(b) *Per* 8. *l.* 1.
(c) *Per* 29. *l.* 3.

equal. Becauſe therefore in the Triangles BAH, HAC, the two Angles S and N are equal, and thoſe at H are both right Angles (*d*), and likewiſe the Side A H is common; therefore the Sides (*e*) BH, CH, as likewiſe the Angles P, T, are equal. In the ſame manner I might ſhew B G, F G to be equal. Therefore B F, C B, which are double to the Equals B G, B H, are alſo equal. In the ſame manner it may be ſhewed that the reſt of the Sides of the circumſcribed Pentagon are equal. It is therefore Equilateral; but it is alſo equi-angled; for ſeeing it hath been ſhewed that the Angles B and C are each of them double to the Equals P and T, they muſt alſo be equal betwixt themſelves. And in the ſame manner of the reſt. We have therefore deſcribed a regular Pentagon about a Circle. Which was the Thing to be done.

(d) *Per* 18. *l.* 3.
(e) *Per* 26 *l.* 1.

In the ſame way any ordinate Figure whatſoever is deſcribed about a Circle, that is, if a like Figure be firſt inſcribed in the Circle.

PROP. XIII XIV. Problems.

TO inſcribe a Circle in a regular Pentagon, and circumſcribe one about it. *Fig.* 11.

Biſect the two Angles of the Pentagon B, C, with the right Lines BN, CS, cutting each other in A. From A draw the Perpendicular A L.

A Circle deſcribed from the Point A, with the Interval AL, touches all the Sides of the Pentagon; and a Circle deſcribed from the ſame Point A, with the Interval A B paſses alſo through the Points E, F, D, C.

Part I. In the Triangles DCA, BCA, becauſe the Sides D C, C A, are equal to B C, C A, by the Hypotheſis, and the Angles P and O are equal by the Conſtruction, thoſe alſo G and I will be equal by 4 *l.* 1. Now, the whole alſo B and D are equal by the Hypotheſis. Wherefore ſeeing the Angle G is half of B by the Conſtruction, I alſo will be half of D. Therefore D is biſected by the right Line D M. For the ſame Cauſe the reſt of the Angles of the Pentagon EF, are biſected, and conſequently all the half Angles are equal among themſelves. Now, let the Perpendiculars be drawn, A M, A S, A N, A R. For becauſe,

because in the Triangles LBA, MBA, the Angles G and BLA are equal to the Angles Q and BMA, by the Construction, and the Side BA is common, AL and AM must

(a) *Per* 26 *l.* 1. be also equal (*a*). In like manner I might shew that the rest of the Perpendiculars, A M, A N, A S, A R, are equal. A Circle therefore from the Center A, passing through L, will likewise pass through M, S, N, R, and because the

* *Per* 16. *l.* 3. Angles at L, M, S, N, R, are right ones by the Construction, * it will touch the five sides of the Pentagon. Which was the first Part.

Part II. In the Triangle C A B, because the Angles O and G have already been shewn to be equal, the sides also

(b) *Per* 16 *l.* 1. A C, A B, must be equal (*b*) and in the same manner, A B, A F, A E, A D, may be proved equal, and therefore a Circle from the Center A, passing through B, must pass also through C, D, E, F. Therefore we have both inscribed a Circle in a Pentagon, and circumscribed one about a Pentagon *Q. E. D.*

[" In the same way, in any regular Figure whatsoever a Circle may be inscribed, and circumscribed about it.

PROP. XV. Problem.

Fig. 13. *IN a given Circle to describe a regular Hexagon.*

Let the Diameter F A B be drawn. From the Center B, through A, describe a Circle, cutting the given one in C, and D. Likewise from the Center F, through A, a Circle cutting the given one in E and G. The six Points B, C, E, F, G, D, connected by right Lines, will give the Hexagon required.

From the Center A, let fall the right Lines A E, A C, A G, A D. It is manifest that the Triangles H, I, M, L, are Equilateral, both in themselves, and with one another

(c) *Per* 1. *l.* 1. (*c*). Then because the Angles C A B, E A F, each of them make one Third of two right Angles (*per Corollary* 12. *p.* 32. *l.* 1.) and therefore do make both together two

(d) *Per Corol.* 1. *p.* 13. *l.* 1. Thirds of two right Angles; it remains (*d*) that E A C is one Third of two right Angles; therefore the Angles E A C, C A B are equal. But the Sides also E A, A C. are equal to the Sides B A, A C. Therefore the Base E C

(per

(*per* 4. *l.* 1.) is equal to the baſe BC, that is to the radius AC by the conſtruction. Wherefore the triangle N is alſo equilateral. And in the ſame manner the triangle K may be ſhewn to be ſo. Becauſe therefore all the ſix triangles H, I, K, L, M, N, are equilateral; it is manifeſt that all the ſides, CB, BD, DG, GF, FE, EC, are equal one to another, and to the radius AC. The Hexagon is therefore equilateral. But it is alſo equiangular, ſeeing each one of its angles EC, BD, GF, conſiſts of two angles of an equilateral triangle. Therefore we have inſcribed a regular hexagon in the circle.

Corollaries.

1. THE ſide of an hexagon inſcribed in a circle, (or a chord of 60 degrees) is equal to the radius, (and conſequently the ſine of 30 degrees is equal to half the radius, *per Corollary* 2. *p.* 3. *l.* 3.)

2. An angle of a regular hexagon is four thirds of one right angle; as conſiſting of two angles of an equilateral triangle, each of which makes two thirds of a right angle.

3. If there be drawn the diameter PS, perpendicular to the other FB; and with the interval of the radius PA, from the center P and S, there may be made ſections in O and Q, in R and T, and in like manner from the centers F and B, make the ſections in G and E, in D and C, the points P, E, O, F, R, G, S, D, T, B, Q, C, connected with right lines, will give a figure of twelve ſides, inſcribed in a circle with one aperture of the compaſſes, which thing is of great ſervice in dialling. *Fig.* 14.

4. From what has been demonſtrated, we may eaſily deſcribe an equilateral triangle in a circle. The diameter FB being drawn, from the center B, through A, deſcribe the arch CAD. The points C, F, D, connected with right lines, will give the triangle ſought. *Fig.* 14.

5. The ſide CXD of the equilateral triangle, cuts off from the diameter BF perpendicular to it, a fourth part thereof BX. For the angles ACX, BCX, ſtanding upon equal arches GD, DB are equal (*per* 29. *l.* 3.) and the ſides AC, CX are equal to the ſides BC, CX. Therefore AX, BX are equal (*a*.) Therefore BX is the fourth part of the diameter BF.

(a) *Per* 4. *l.* 1.

Scholium 1. *Problem.*

Fig. 13. YOU may raiſe a regular hexagon upon a right line
* *Per* 1. BC thus. Make an * equilateral triangle CAB
l. 1. upon the given line CB. From the center A through B and C deſcribe a circle. This will contain an hexagon upon the given right line CB. The thing is manifeſt from the propoſition, and *Corollary* 1.

Theorem.

THE ſquare of a ſide of an equilateral triangle is treble to the ſquare of the ſemidiameter of a circle in which it is inſcribed, and is to the ſquare of the whole diameter, as 3 to 4.

Fig. 14. Let there be drawn the ſemidiameter AD. The ſquare of FD is equal to FAq + DAq + the rectangle FAX twice taken (*per* 12. *l.* 2.) But the rectangle FAX twice taken, is equal to the ſquare of the ſemidiameter FA or
(b) *Per* DA: [For becauſe AX, XB (*b*) are equal, the rectangle
Corol. 5. FAX twice taken, is equal to the two rectangles which
foregoing. are under FA, AX, and under FA and XB, that is,
(c) *Per* 1. equal to the rectangle FAB (*c*); that is, equal to FAq.]
l. 2. Therefore FDq is treble to FAq or DAq, the ſquare of the ſemidiameter.

Now, becauſe the ſquare of the whole diameter is qua-
(d) *Per* druple of the ſquare of FA, the ſemidiameter (*d*), it is
Corol. 3. manifeſt that the ſquare of FD is to the ſquare of the
Prop. 4. diameter, as 3 to 4.
l. 2. Hence it follows, that a ſide of an equilateral triangle is to the diameter, as the ſquare root of 3 is to 2, the ſquare root of 4; and therefore that thoſe lines are incommenſurable.

PROP.

PROP. XVI. Problem.

ſcribe a reg[illegible]lar quindecagon in a circle. *Fig.* 15.

[illegible]he circle A C, the ſide of a pentagon (*a*,) and (a) *Per* 11. *l.* 4.
[illegible]de of an equilateral triangle (*per* Corol. 4.
[illegible] biſect the arch C D in E. C E is the ſide of
[illegible]cagon, or fifteen-angled figure ſought.
[illegible]e whole circumference be ſuppoſed to be 15,
[illegible]C will be 3, and the arch A D 5, and therefore
C D 2, and conſequently C E 1.

Corollary.

BY this method innumerable regular figures may be inſcribed in a circle. For if A C, A D, the ſides of two regular figures be inſcribed in a circle, the difference of the arches C D will contain ſo many ſides of a new regular figure, as are the units whereby the denominators of the former differ one from another. But by the denominator of the new figure is had, if the denominators of the former be multiplied one by the other. *Fig.* 15.

As if A D be the ſide of a ſquare, and A C of a decagon, the difference of the denominators is 6. Therefore the arch C D contains 6 ſides of a new figure. But the new figure is of 40 ſides. For the denominators 4 and 10, multiplied one by the other, make 40.

Scholium.

THERE hath not yet been found out the art by which regular figures of 7, 9, 11, 13, 17, &c. ſides may be inſcribed in a circle, by a pair of compaſſes and a rule only; forasmuch as that inſcription of figures depends upon the diviſion of the circumference into any given parts, which thing is lacking: But if the circumference of a circle be divided into 360 Parts, you may, in a mechanical way, inſcribe any regular figures whatſoever in it, after this manner.

Problem 1.

Fig. 15. DIVIDE 360 degrees (that is the whole ci by the denominator of the polygon to (*e. g.* a nonangle.) Make at the center the ang so many degrees as are the units of the qu said division. A K shall be the side of the r gure, which is required to be inscribed in th

Problem 2.

Fig. 15. BUT upon a given right line you may regular figure whatsoever by the help of ing table:

A right angle is to the angle of the figure.

		Difference.
In a Pentagon, as - - - -	5 to	6—1
In an Hexagon, as - - -	3 to	4—1
In an Heptagon, as - -	7 to	10—3
In an Octagon, as - - -	2 to	3—1
In a Nonagon, as - - -	9 to	14—5
In a Decagon, as - - -	5 to	8—[illegible]
In an Undecagon, as - -	11 to	18—7
In a Duodecagon, as - -	3 to	5—2

Fig. 15. Let a regular heptagon be to be inscribed upon the given right line, XB. From the center X, with the interval X B, describe a circle, from which cut off the quadrant B O. See in the table what is the proportion of a right angle, to the angle of an Heptagon: You will find it to be as 7 to 10, and the difference 3. Divide the quadrant therefore into seven equal arches, so many of which add to it from O to N, as the difference hath units. Through three points, B, X, N, describe (*per* 5. *l.* 4.) a circle. This contains an heptagon of the given right line X B.

The table was made by means of *Theorem* II. in the *Schol.* upon *p.* 32. *l.* 1. by which is found the number of right angles, which the angles of any right-lin'd figure make; which number being divided by the denominator of the figure, gives the denominator of the proportion of the angle of the figure to a right one.

Now,

Now, because hitherto many things have been propounded concerning regular figures, let the following famous theorem of *Proclus* close this book.

Theorem.

ONLY three regular figures, to wit, six equilateral triangles, four squares, and three hexagons, can fill a space; that is, constitute one continued superficies. *Fig.* 13. Which is thus demonstrated. That some regular figure, often repeated, should be able to fill a space; it is required that the angles of many figures of that kind being disposed about one point, should make just four right ones; for just so many right angles may be placed about one point, as appears from *Corollary* 3. *Prop.* 13. *l.* 1. As for example, that equilateral triangles should fill a space, it is required that so many angles of such triangles N, M, L, K, I, H, being disposed about the point A, should make just four right ones. But six angles of an equilateral triangle do make four right ones; (for one makes two thirds of one right one *, and therefore six of them make twelve thirds of one right one, that is, four right ones:) Likewise the four angles of a square make four right ones, as is manifest; likewise three angles of an hexagon; for one maketh four thirds of one right angle, (*per Corollary* 2. *p.* 15. *l.* 4.), and therefore three of them do make twelve thirds of one right angle, that is, four right ones. Therefore, &c.

* *Corol.* 12. *p.* 32. *l.* 1.

But that no other figure beside these can do this, will manifestly appear, if its angle being found, as above, you shall multiply the same by any number whatsoever; for the angles will always either fall short of, or exceed four right ones.

THE

ELEMENTS

OF

EUCLID

BOOK V.

THIS fifth book of elements is altogether neceſſary for demonſtrating the propoſitions of the ſixth Book. The Doctrine which it containeth is almoſt in continual uſe. The way of reaſoning from geometrical proportion is moſt ſubtil, ſolid and brief. This method of reaſoning, as a kind of mathematical logic, geometry, arithmetic, muſic, aſtronomy, ſtatics, and all the other parts of the mathematicks, make eſpecial uſe of: Foraſmuch as they almoſt wholly depend upon proportions connected together one with another; and are wont to borrow their ways of reaſoning concerning proportionals from this fifth book. Practical geometry, which conſiſts in the meaſuring of lines, figures and ſolids, is for the moſt part derived from the doctrine of proportions. There is not a rule in arithmetic, but what may be demonſtrated from the propoſitions of this fifth book, without the help of the 7th, 8th, and 9th books, which treat profeſſedly of numbers. We may fitly call the muſic of the ancients, geometrical proportions applied to tuneful ſounds; which ſame thing you may well nigh ſay concerning ſtatics, which are converſant about the weights of bodies. To comprehend the whole matter in few words: If you take away the doctrine of proportion from the mathematics, you will leave almoſt nothing which is excellent, or greatly to be accounted of.

Scholium.

Scholium.

" There is no mathematician who is ignorant of how " great importance in geometry the knowledge of pro- " portion is; for it is the very marrow, as it were, of the " mathematical sciences. And the various ways of rea- " soning concerning proportionals, are both most useful, " and most certain; neither can we without them move " one step.

" But then I reckon that this doctrine is congenite in " men's minds with common reason itself; and that the " various ways of reasoning concerning proportionals, " which *Euclid*, by much winding and going about, deli- " vers in this whole book, do not so much need demon- " stration, properly so called, as illustration and examples. " And I am altogether of opinion, that those who take in " hand to deliver this most easy doctrine by a long circuit " of propositions, do involve a thing in itself most clear, " in a certain cloud, and render it far more difficult. The " sum of the matter I will open in a few words. It is a " thing easily known, that four quantities are then pro- " portional, or that the analogies are then alike, when the " first quantity contains the second, as often as the third " contains the fourth; or when the first is as often con- " tained by the second, as the third is by the fourth. So " 16 : 8 :: 4 : 2. And 3 : 9 :: 4 : 12. Here are like, " or the same proportions; because in the former exam- " ple, the consequents 8 and 2 are contained twice in their " respective antecedents; and so the proportion of the an- " tecedents to the consequents, is double. And in the " other example, the proportions are also alike, because " the consequents 9 and 12 do contain their respective an- " tecedents three times; and so the proportion of the an- " tecedents, to the consequents, is subtriple. (Nor is " there any proportion of commensurable quantities which " may not be expressed by certain numbers: nor indeed of " incommensurables, which may not be expressed by num- " bers infinitely approaching nearer and nearer unto the " true one.) Furthermore, from what hath been said it " appears, that like proportions, whatsoever they are, " may be expressed not only by divers numbers, but also " by the same. Thus 2 to 1 designs as well the propor-

" tion

" tion of 16 to 8, as of 4 to 2; 1 to 3 no less expresseth " that of 4 to 12, than that of 3 to 9, as is most manifest. " Supposing therefore four quantities to be proportional, " A : B : : a : b; it is enquired in this book after how " many like manners the terms of these like proportions " may be changed, and ordered amongst themselves. " So that the emerging proportion on both sides may be " still alike. And it may be answered, that it may be " done after all the ways and manners possible; for see-" ing the proportion of A to B, and that of a to b are " alike, both of them may be expressed by the same num-" bers after this manner, A : B : : 9 : 3, and a : b : : 9 : 3. " And consequently all the proportions emerging on both " Sides, either by *Alternating* the terms, or by *Inverting* " them, or by *Compounding*, or *Dividing*, or *Converting*, " or *Mixing* them, may be expressed by the *very same* " numbers; and consequently the same proportion will " always be kept on both sides. As for example sake: " A+B : B : : a+b : b, because 9+3 : 3, expresseth the " same proportion; which is *Composition*. The same is " to be said of all the ways of changing the terms. " Therefore let beginners observe this one thing, that " proportions, which are on both sides the same, be ever " changed and ordered in the very same manner. And " then there will be no room to question, whether the " proportions which arise on both sides be alike or no. " It is indeed a thing to be wondered at, that no one of " those who have hitherto compiled elements of geome-" try, have made use of this most easy method of stating " the equality of proportions, for the illustrating of this " fifth book about the doctrine of proportions. Take " therefore the primary ways which geometry makes use " of, in reasoning concerning like proportions, as they " are digested into this short table.

Let

Let it be	$A : B :: a : b :: 9 : 3$
Then it will be by	
Alternating	$A : a :: B : b :: 9 : 9 :: 3 : 3$
Inverting	$B : A :: b : a :: 3 :: 9$
Compounding	$A+B : B :: a+b : b :: 9+3\,(12) : 3$
Dividing	$A-B : B :: a-b : b :: 9-3\,(6) : 3$
Converting	$A : A+B :: a : a+b :: 9 : 9+3\,(12)$
or	$A : A-B :: a : a-b :: 9 : 9-3\,(6)$
Mixing	$A+B :: A-B :: a+b : a-b :: 9+3 : 9-3.$
Ex æquo	$A : B :: a : b$, & $B : C :: b : c$, *then* $A : C :: a : c.$
	$9 : 3 :: 9 : 3 \quad 3 : 1 :: 3 : 1$, *then* $9 : 1 :: 9 : 1.$
Ex æquo	$A : B :: a : b$, & $B : C :: r : a$, *then* $A : C :: r : b.$
perturbate	$8 : 3 :: 8 : 3$, & $3 : 12 :: 2 : 8$, *then* $8 : 12 :: 2 : 3.$
Or thus,	$8 : 3 :: 16 : 6$, & $3 : 2 :: 24 : 16$, *then* $8 : 2 :: 24 : 6$:
	$a : b :: e+a : a+b$ & $b : e :: a+b : a+e$ *then* $a : e :: a+b : e+b.$

" He therefore that is expert in theſe ways of reaſoning " concerning proportionals, and knows how to bring them " into uſe upon occaſion, will ſeldom ſtand in need of the " particular propoſitions of the fifth book. Only two of " them, which yet are almoſt axioms, may not improperly " be inſerted and illuſtrated by example, in way of *Appen-* " *dix*, becauſe of the frequency of their uſe in all parts " of the mathematics; which therefore ſhall be done af- " ter the definitions.

DEFINITIONS.

1. AN *Aliquot* part of magnitude, is that which being ſo many times more or leſs repeated, doth meaſure, or is juſt equal to the magnitude. An *Aliquant* part is that which doth not meaſure it.

The length of one foot is an *Aliquot* part of the length of 10 feet, becauſe being ten times repeated, it meaſures it. But the length of 4 feet is an *Aliquant* part of a line of 10 feet, becauſe being ſo many times repeated, to wit, twice, it falls ſhort of it, but being thrice repeated, it exceeds it.

2. One magnitude is ſaid to be a *Multiple* of another, when the leſſer meaſures the greater, and conſequently is an

an *Aliquot* part thereof; or when the greater contains the leſſer ſo many times preciſely.

3. Proportion is the mutual reſpect, as to quantity, of two magnitudes of the ſame kind.

Therefore there are in all proportions two terms, of which that is called the antecedent which is firſt named, or which is named in the nominative caſe; the other the conſequent.

When the antecedent and the conſequent are equal, it is called proportion of equality; when they are unequal, proportion of inequality.

4. Rational proportion, is that which is betwixt commenſurable magnitudes, and may be expreſſed in numbers. Irrational proportion, is that which is betwixt quantities incommenſurable, and cannot be explicated by any numbers.

Moreover, commenſurable quantities, are thoſe which ſome common meaſure meaſureth; incommenſurable, thoſe which cannot be meaſured by any common meaſure.

Fig. 1. *l.* 5. 5. Two proportions (that of A to B, and that of C to F) are alike, equal or the ſame; when the antecedent of one (A) doth equally, or in the ſame manner, that is, neither more nor leſs, contain its conſequent (B) as the antecedent of the other (C) contains its conſequent (F.)

Fig. 2. Or when the antecedent of the one (A) is ſo often contained in its conſequent (B) as (C) the antecedent of the other is in its conſequent (D.)

Fig. 3. 6. Two proportions are unlike, or one is greater than the other, when the antecedent of one (I) doth more contain its conſequent (L), than the antecedent of the other

Fig. 4. (O) doth contain its conſequent (Q;) or when the antecedent of one is leſs contained in its conſequent, than the antecedent of the other in its conſequent.

7. Like or *ſimilar* parts are thoſe which are equally or in the ſame manner contained in their wholes; ſo that what ſort of part one is of its whole, ſuch a part the other is of its whole. Which thing indeed is nothing elſe, but that the parts bear the ſame proportion to their wholes.

Aliquot parts are alike, which do equally meaſure their wholes, as if each of them be one third or one tenth, &c. of its whole.

Fig. 6. 8. Magnitudes (A, B, C, D) are ſaid to be *continually* proportional when the middle terms (B, C) are taken

twice;

twice; that is, when they are each of them a consequent in respect of the foregoing, and an antecedent, in respect of the following.

We thus pronounce continual proportions. A is to B, as B to C; and B is to C, as C is to D. And so on.

9. Magnitudes are *discretely* proportional when no term is twice taken.

Discrete proportions we thus pronounce: A is to B, as C to F. When they are more than three proportional magnitudes if they be said to be proportional, they are always understood to be *discretely* so. *Fig.* 1.

10. When the magnitudes (A, B, C, D) are continually proportional, the first (A) is said to have to the third (C) a duplicate proportion of that which it hath to the second (B:) And the first (A) is said to have to the fourth (D) a triplicate proportion of that which the same first hath to the second (B:) And so forwards. *Fig.* 6.

[" If one triplicate proportion be equal to another duplicate proportion, the latter simple proportion shall be sesquiplicate, or one and a half of the former simple proportion. Let A, B, C, D, be ∺; and a, b, c, ∺; and let A the first in the former analogy be unto D the fourth; as (a) the first in the second analogy is to (c) the third; I say, that (a) is to (b) in a proportion, which is one and a half of that which A is in to B. For let F be a middle proportional between B and C: Or, which is the same thing, betwixt A and D. Because of the equality of the proportion of A to D, and (a) to (c) and the middle proportions on both sides F and (b;) it will be A : F : : a : b. But the proportion of A to F is compounded of the entire proportion of A to B, and of the proportion of the same B to F halved; and consequently the proportion of (a) to (b) which is equal to that of A to F, contains the entire proportion of A to B, and also the same halved, to wit, the proportion of B to F. But the whole proportion, with its half, is a sesquiplicate or sesquialteral proportion, or that which is one and a half of the other. (a) Therefore is to (b) in a proportion sesquiplicate of that of A to B. So in astronomy, since the cubes of the distances of the planets from the sun bear that proportion one to another, which the squares of their periodical times bear; so that the triplicated proportion of the distances, is the

" same

" ſame with the duplicate one of the periodical times; " it is wont to be ſaid, that the periodical times are in a " ſeſquiplicate or ſeſquialteral proportion of their diſtances " from the ſun.]

Fig. 1. 11. Antecedent magnitudes are ſaid to be homologous, or like to antecedent, and conſequent to conſequent magnitudes. As if A is to B, as C to F; A, C, and B F, are homologous quantities.

XII. If a ſet of pairs of quantities contain every one the ſame proportion, that is the very proportion alſo which the ſum of all the antecedents bears to the ſum of all the conſequents.

$$\frac{20+6+8+18+14=66}{10+3+4+9+7=33}$$

XIX. If parts be as wholes, the remainders will be alſo in the very ſame proportion.

If 30 be to 20, as 3 to 2; 27 will be to 18 alſo as 30 to 20, or as 3 to 2.

TH

THE ELEMENTS OF EUCLID.

BOOK VI.

THE doctrine of proportions, which was generally set forth in the fifth book, is applied in the sixth, to plain figures. And those things which are delivered in this book are so necessary to be known, that without them no man can penetrate into the secrets of geometry, and reap the sweet fruits of the mathematics. Each proposition deserves to have an encomium annexed; so great is the utility of all.

"This sixth book, as hath been said, begins to apply "that excellent doctrine concerning geometrical propor-"tion, which was just before delivered, to divers, and "those certainly, most notable uses; and beginning with "triangles, the most simple of figures, searches out their "sides and areas, as they answer to one another in a cer-"tain proportion. Then it defines proportional lines, "and the proportional augmentations, or diminutions of "figures; and shews in what manner we may either en-"crease or diminish them according to any proportion gi-"ven. It opens likewise the golden rule, or rule of propor-"tion, the very chief of all arithmetic; and demonstrates "that in a rectangle triangle, not only the square, but also "the pentagon, hexagon, and in general, every angular "polygon, which is described by the hypothenuse, is equal "to the squares, pentagons, hexagons, or any regular "polygons whatsoever, that are described by the two sides.

"It

"It alſo propounds moſt eaſy and certain principles for "meaſuring as well ſolids, as lines and ſurfaces, "which are of very great uſe in all parts of the mathema- "tics.

DEFINITIONS.

1. LIKE or ſimilar figures, are thoſe which both have all the angles equal, each to each other reſpectively, and the ſides which are oppoſed to the equal angles, or which are betwixt them, or which are about the equal angles, (for they come all to one) proportional.

Fig. 7. l. 6. As the triangles X, Z, will be ſaid to be like, or ſimilar, if the angle A be equal to the angle F, and the angle B equal to the angle I, and conſequently the angle C equal to the angle L: And moreover, if A B be to F I, as B C to L I; and B C is to L I, as C A is to L F; and C A to L F, as A B to F I; by comparing always the ſides oppoſite to the equal angles. In the ſame manner the likeneſs of all right-lined figures may be explained.

Fig. 29. 2. Reciprocal figures, are when the antecedent and conſequent terms of the proportions appear on both ſides.

As in the parallelograms X Z,
If A C be to C B,
As F C is to C L.

The antecedents here are A C and F C, of which there is one in both figures; and the conſequents are C B, and C L; of which likewiſe there is one in each figure. And therefore the parallelogram X, Z, are called reciprocal. Underſtanding the ſame of other figures.

3. The altitude of a figure is the perpendicular let fall from the top to the baſe. This is with *Euclid* the fourth definition.

Fig. 1. As the altitude of the triangle A B C is the perpendicular AQ, which falls from the top upon the baſe B C, either within the triangle or without, upon the baſe protracted. Now the baſe and top are aſſumed at pleaſure.

4. Like arches of circles are thoſe which have the ſame proportion unto their whole circumferences.

As if each of them be a third or fourth part, &c. of that circumference.

PRO-

PROPOSITION I. Theorem.

TRiangles (*ABC, DEF*) *and parallelograms* (*AOPC, DQRF*) *which are betwixt the same parallels, or have the same altitude, have the same proportion betwixt themselves as their bases,* (*AC, DF.*) *Fig.* 2.

Upon this theorem the whole sixth book depends, yea, whatsoever any where is demonstrated about figures by proportions, whether plane or solid.

Let there be taken any aliquot part of the base DF; *e.g.* DG one third, and let the right line GE be drawn: The triangle DEG will likewise be one third part of the triangle DEF, as is gathered from 38. l 1. Wherefore DG, and the triangle DGE are like aliquot parts of their consequents *. Then let there be taken away DG from the base AC, as often as it can, as suppose six times, and let the right lines HB, IB, KB, LB, MB, NB, be drawn. Because the lines CH, HI, *&c.* are each of them equal to DG, the six triangles CBH, HBI, *&c.* are each of them (*a*) equal to the triangle DEG. Therefore as often as DG is contained in the antecedent AC, so often is the triangle DEG contained in the triangle ABC. By the same reasoning it may be shewed, that the like aliquot parts whatsoever of the consequents (the base DF, and the triangle DEF) are in an equal number contained in the antecedents (the base AC, and the triangle ABC:) Therefore as the base AC, is to the base DF; so is the triangle ABC, to the triangle DEF. *Q. E. D.*

* *Per Def.* 7. *l.* 5.

(a) *Per* 38. *l.* 1.

But now, because the parallelograms AP, DR are (*b*) double to the triangles ABC, DEF, they also will be as their bases.

(b) *Per* 41. *l.* 1.

Corollary.

THE triangles (ABC, FIL) and the parallelograms which have equal bases (AC, FL) or the same, have that proportion one to another, which their altitudes (BO, IQ) have. *Fig.* 3.

For

For let QS, OR, be made equal to the equal bases (FL, AC;) QS, OR will then be equal. Draw SI, RB. If in the triangles OBR, QIS, you take BO, IQ for the bases, OR, QS, will be their altitudes; which seeing they are equal, the triangles OBR, QIS

(c) *Per* 1. *l.* 1. (*c*) will be betwixt themselves, as their bases BO, IQ. But because by the construction, OR is equal to AC,

(d) *Per* 38. *l.* 1. and QS equal to FL, the triangles OBR, QIS, are *(d)* equal to the triangles ABC, FIL. Therefore the tri-

Fig. 50. angles ABC, FIL, are also as BO is to QI.

Corollary (2.) "Hence a trapezium ABCD, whose "sides AD and BC are parallel, may be divided into any "equal parts whatsoever. For let CE be made equal "to AD. Because of the equality of the angles verti-

(e) *Per* 15. *l.* 1. "cally opposite *(e)* AFD, EFC, and of the alternate "angles *(f)* DAF, FEC, and ADF, ECF, and the

(f) *Per* 27. *l.* 1. "equality of the bases AD, CE, by construction, the "triangles ADF, FCE (*g*,) are equal; and therefore the

(g) *Per* 26. *l.* 1. "triangle ABE is equal to the trapezium ABCD. "Therefore the base BE being divided into any equal parts "whatsoever; as for instance, three, BG, GR, RE the "triangles ABG, AGR, ARE, shall each of them "be one third part of the trapezium. *Q. E. I.*

PROP. II. Theorem.

Fig. 4. *If to one side of a triangle (as BC) there be drawn (FL) a parallel, this cuts the sides proportionally, that is, (AF) will be to (FB) as (AL) to (LC.)*

And if the right line (FL) cuts the sides (BA, CA) proportionally, it will be parallel to the other side (BC.)

Part I. Let BL, CF be drawn, because FL is supposed parallel to BC, the triangles FBL, LCF having the

(a) *Per* 37. *l.* 1. same base are (*a*) equal. Therefore the triangle X hath the same proportion to both; now the triangle X is to

(b) *By the foregoing.* the triangle FBL, as the same triangle X is to that LCF. But the triangle X is to the triangle FBL

(c) *By the same.* (*b*,) as AF is to FB; and the triangle X is to that LCF as AL (*c*) is to LC. Therefore also AF is to FB, as AL to LC. *Q. E. D.*

Part II. As A F is to F B, so is the (*d*) triangle X to the triangle F B L: And as A L is to L C, so is the same triangle X to the triangle L C F. Now A F is supposed to be to F B, as A L is to L C. Therefore the triangle X is to the triangle F B L, as the same X is to L C F. Therefore the triangles F B L, L C F are equal. Therefore seeing they have a common base F L, the lines F L, B C, are (*e*) parallel. *Q. E. D.*

(d) *By the foregoing.*

(e) *Per* 39. *l.* 1.

Corollary.

IF unto (B C) one side of a triangle there be drawn more parallels (I O, F L) all the segments of the sides will be proportional. *Fig.* 5.

Let F Q be drawn parallel to A C. The right line F S, S Q, are equal (*f*) to L O, O C. But B I is to F I, as Q S to S F (*a*.) Therefore B I is also to I F, as C O to O L.

(f) *Per* 34. *l.* 1.

(a) *Per* 2. *l.* 6.

PROP. III. Theorem.

IF a right line (B F) which bisects an angle of a triangle, doth also cut the base (A C), the segments of the base (A F, F C) will have the same proportion betwixt themselves as the sides (A B, B C) have. *Fig.* 6.

And if the parts of the base (A F, F C) have the same proportion betwixt themselves, as the other sides (A B, C B) the line (B F) which cuts the base, bisects the opposite angle (A B C.)

Part I. Draw from C B until B L be equal to B A; and join A L. Because in the triangle Z, the sides L B, A B, are equal, the angles also (*b*) L and O are equal. Because therefore the external angle A B C is equal to the two internal ones (*c*) L, O, the angle I, which by the hypothesis is half A B C, will be equal to the angle L. Therefore A L, F B (*d*) are parallel. Therefore in the triangle A C L, A F is to F C (*e*) as L B (that is, A B) is to B C. *Q. E. D.*

(b) *Per* 5. *l.* 1.

(c) *Per* 32. *l.* 1.

(d) *Per* 29. *l.* 1.

(e) *Per* 2. *l.* 6.

Part II. Protract CB again, until BL be equal to BA. Because AF is supposed to be to FC, as AB (that is, LB) is to BC; AL, FB (*a*) are parallel. Therefore the external angle I is equal to the internal one L (*b*); and the alternate Q equal to the alternate O. But because LB, AB, are equal, the angles L and O (*c*) are equal. Therefore I and Q are also equal. Therefore ABC is bisected. *Q. E. D.*

(a) *Per* 2. *l.* 6.
(b) *Per* 27. *l.* 1.
(c) *Per* 5. *l.* 1.

PROP. IV. Theorem.

TRiangles which are equiangular to one another, are like or similar, that is, have their sides also (a) *that are opposite to the equal angles proportional.*

(a) *Per Def.* 1. *l.* 6.

Fig. 7. In the triangles XZ, let the angle A be equal to the angle F, and the angle C to the angle L, and the angle B to the angle I; I say, that AB is to FI, as AC is to FL; and AC is to FL, as CB is to LI; and CB is to LI, as BA is to FI.

Fig. 7, 8. *Demonst.* If the angle F be laid upon its equal A, the sides FI, FL, will fall upon the sides AB, AC. And because the external angle AIL is by the hypothesis equal to the internal B (*b*), therefore (*c*) IL, BC, are parallel. Therefore BI is to IA (*d*) as CL to LA. Therefore by compounding, BA is to IF, as CA to LF. And if the angle L be laid upon the angle C, it will be shewed in the same manner, that AC is to FL, as BC is to IL; and if the angle I be laid upon the angle B, it will be shewed in the same manner, that BC is to IL as AB to FI. The proposition therefore is proved.

(b) *Fig.* 8. *alone.*
(c) *Per* 29. *l.* 1.
(d) *Per* 2. *l.* 6.

Corollaries.

Fig. 8. 1. IF in a triangle a line LI be drawn parallel to one side BC, the triangle LFI will be like to the whole CBF; and consequently CF will be to LF, as BC to LI.

For because LI, BC, are parallel, the external angles FIL, FLI will (*per* 27. *l.* 1.) be equal to the internal ones B and C: But F is common to both triangles. Therefore

fore they are equiangular. Therefore the ſides C F, L F oppoſite to the equal angles B and F I L (*a*) are proportional to the ſides B C, L I, which are oppoſed to the common angle F. (a) *By the foregoing.*

2. If in a triangle a right line B F, drawn from the oppoſite angle B, doth cut the parallels A C, L O it cuts them proportionally. *Fig.* 9.

For by *Corollary* 1. A F is to L I, as F B is I B; and F C alſo is to I O, as F B is to I B. Therefore A F is to L I as F C to I O. Therefore by changing, A F is to F C, as L I to I O.

[3. From *Corollary* 1. "We learn to find the height of a tower, or any elevated point, by only the ſhadow "of a ſtaff. Fix the ſtaff F L perpendicularly upon the "Ground in that place where the ray of the ſun X B A, "that terminates the ſhadow of the tower B Z, may alſo "paſs through L. There will be in the triangle A Z B, "the line F L, parallel to the height of the tower Z B. "Whence as A F, the diſtance of the ſtaff from the "point of the ſhadow, is to F L, the length of the ſtaff: "ſo is A Z, the diſtance of the tower, from the point of "the ſhadow, to Z B, the height of the tower. And "becauſe the three firſt terms are eaſily had by meaſuring "the fourth alſo, *i. e.* the height of the tower is had "alſo. *Q. E. I.* *Fig.* 51.

4. "From this alſo incomparably uſeful propoſition, "we may deduce that famous theorem of *Ptolomy*; to "wit, that in every quadrilateral inſcribed in a circle, the "rectangle of the diagonals A C × B D is equal to the "two rectangles of the oppoſite ſides, A B × C D and "AD×BC. For let the angle BAE be made equal to the "the angle C A D. Becauſe the angles B A E, C A D, "are equal by conſtruction, the angles A B E, C A D, "ſtanding upon the ſame arch A D, are * equal; there- "fore the triangles B A E, C A D, are alike. And A C : "C D : : A B : B E; and conſequently † the rectangle of "the extremes A C × B E is equal to the rectangle of the "means C D × A B. In like manner, becauſe the angle "E A D is equal to the angle B A C by conſtruction, "and the angles A D E, A C B, as ſtanding upon the "ſame arch A B, are equal: The triangles A D E, "A C B, will be like; and A D : D E : : A C : C B. And "therefore the rectangles of the extremes A D × C B is *Fig.* 52.

* *Per* 21. *l.* 3.

† *Per* 16. *l.* 6.

"equal to the rectangle of the means D E×A C. But the
(a) *Per* 1. "rectangles A C×B E, and A C×D E, are equal (a) to the
l. 1. "rectangle A C×B D. Therefore the rectangles A B×D C, "and A D×B C, which are made by the opposite sides, "are equal to the rectangle A C×B D, which is made by "the diagonals. *Q. E. D.*

PROP. V. Theorem.

Fig. 10. *IF two triangles have all their sides proportional each to each, they shall also be mutually equiangular.*

That is, if A B be to R F, as A C to R Q; and as A C is to R Q, so is C B to Q F; and as C B is to Q F, so is A B to R F; I say that the angles opposite to the antecedents, are equal to the angles opposite to the consequents; to wit, C to I, and B to F, and A to O.

Ang.	Antec.	Conseq.	Ang.
C	A B	R F	I
B	A C	R Q	F
A	C B	Q F	O

Make X and Z equal to A and C; and let the sides
(a) *Per Cor.* meet in N. The angles B and N will (*a*) be also equal.
9. *p*. 32. Because therefore the triangles P, T, are equiangular,
l. 1. A B (by the foregoing) will be to R N, as A C to R Q. But by the hypothesis, A B is to R F, as A C to R Q. Therefore A B is to R F, as the same A B is to R N. Therefore R N, R F, are equal. In the like manner, I might shew that Q N and Q F are equal. Therefore the triangles T, S, are equilateral to each other. Therefore the angles, I, F, O, are equal (*per* 8. *l*. 1.) to the angles Z, N, X, that is, by the construction to the angles, C, B, A. *Q. E. D.*

PROP. VI. Problem.

Fig. 10. *IF two triangles (P, S,) have an angle (A,) equal to one angle (O,); and the sides (A B, AC, R F, R Q,) which contain the equal angles propotional; the triangles will be like.*

Let

Let X and Z be made equal to the angles A C, and the sides meet together in N. Therefore the angles B and N will * be also equal. Then it may be shewed, as in the foregoing, that R F, R N, are equal. But R Q is common to both triangles S, T. The angles also O and X are equal, because they are equal to the same A, the one X by the construction, and O by the hypothesis, Therefore (c) I and F are likewise equal to Z and N. Therefore the triangle S is equiangular to the triangle T; that is, by the construction, to the triangle P. Therefore S, P are like *(per* 4. *l.* 6) *Q. E. D.*

(*) *Per Corol.* 9. *p.* 32. *l.* 1.

(c) *Per* 4. *l.* 1.

PROP. VII.

IS *scarce of any use.*

PROP. VIII. Theorem.

IN a rectangle triangle, the perpendicular (B C) let down from the right angle to the base, cuts the triangle into parts like to the whole, and betwixt themselves. *Fig.* 11.

In the triangles A B F and I, the angle F is common, but the angles A B F and X are, by the hypothesis right ones, and consequently equal. Therefore the other angles A and O are (*b*) also equal. Therefore * the triangles A B F and L are like: In the same manner the triangles A B F and R may be shewed to be equal, and the angle I equal to the angle F. From which it is now manifest, that R and L also are like, seeing the angles I and F, O and A, U and X, are equal. *Q. E. D.*

(b) *Per Corol.* 9. *Prop.* 32. *l.* 1.

* *Per* 4. *l.* 6.

Corollaries.

FIRST, B C is a mean proportional betwixt A C and C F.

For seeing there be in the triangle R and L,

equal angles, I, F } { equal angles, A, O
sides opposed, A C, C B } { sides opposed, C B, C F.

(a) *Per* 4. *l.* 6. It is manifest (*a*) that A C : C B : C F ÷

2. B F is a mean proportional betwixt A F, and C F. Likewise A B, a mean betwixt F A, and C A.

For in the triangle A B F and L,

equal angles, A B F, X } { equal angles, A, O
sides opposed, A F, B F } { sides opposed, B F, C F.

(b) *By the same.* Therefore A F (*b*) : B F : : B F : C F. Likewise because in triangle A B F and in R there be

equal angles, A B F, V } { equal angles, F I
sides opposed, A F, A B } { sides opposed, A B, A C

It will be again A F : A B : A C ÷

Fig. 11. 3. "Hence we learn to measure an inaccessible line, "one term whereof is accessible. Let the inaccessible "line be C F. Let their be raised from the point C the "perpendicular C B: And to any point of this perpendi-"cular, as B, let there be applied a square, or any right "angle A B F; so that in looking along the line B F, the "point F, and along the side B A, the point A may be ob-"served. Let the accessible line A C be measured, and "from the following analogy the inaccessible C F will "be made known. A C : C B : : C B : C F. Let the square "then of the line C B be divided by the line A C, and "the quotient (*c*) will give the sought line C F. *Q. E. I.*

(c) *Per Corol.* 3. *p.* 17. *l.* 6.

PROP. IX. Problem.

Fig. 12.

TO *divide a given line (A B) according to a given proportion (F I to I L.)*

Let the infinite line A Z be drawn. From which take A Q, Q R, equal to F I, I L. From R draw R B. Parallel to this, draw Q C from Q. I say, the thing is done.

It is manifest from *Proposition* 2. *L.* 6.

PROP.

PROP. X. Problem.

TO divide a given line, as (A B) in like manner as another given one (A I) hath been divided (in F, C.) *Fig.* 13.

Let the right line I B join the extremities of the two lines. Draw parallels to this from the points, F, C, which may meet the right line, that is to be cut, A B in L and Q. I say the thing is done.

This is manifest from the *Corollary* of *Proposition* 2. *l.* 6.

[" Or thus, if the cut line I A be greater than that *Fig.* 53.
" which is to be cut, B Q, let three circles touching one
" another, be described with the diameter I F, I C, I A;
" and let the subtense B Q be fitted from the point I to
" the circumference of the greatest circle: the two lesser
" circles will cut the line B Q in the point L, P, in the
" proportion § of the sections of the diameter I A. If § *Per Cor.*
" the line I A be cut into four parts, four circles are to 4. *p.* 31.
" be drawn; if into five, then five circles; and so infinite- *l.* 3.
" ly.]

Scholium.

FROM this proposition we learn to cut a right line given *Fig.* 13. into any equal parts whatsoever. Let an infinite right line make any angle with the right line to be cut, A B; from which take, with a pair of compasses, so many equal parts, A C, C F, F I, as you would divide A B into. Draw the right line I B, and the parallels to it, F L, C Q. I say the thing is done.

We may do the same thing otherwise, and more easily *Fig.* 14. after *Maurolycus*, in the manner following. Let A B be to be trisected or divided into three equal parts. Draw the infinite line I X parallel to A B, above or below it. From I X, if it be below A B, take with a pair of compasses three equal parts, I Q, Q R, R S, which together may be greater, than A B; but lesser, if I X is above. Through I and A, as likewise through S and B, draw right lines which may meet together in C. From C to Q and R draw right

right lines: These will trisect the given line AB. The demonstration appears from *Corollary* 2. *Prop.* 4.

Fig. 15. Again, with *Maurolicus*, we may otherwise obtain the same thing, to wit, thus: Let AB be to be quadrisected. Draw the infinite line AX, and BZ also an infinite line parallel to it. From these take with the compasses equal parts AL, LO, OQ, and BV, VS, SR, in each fewer parts by one than are required in AB; then let there be drawn the right lines, LR, OS, QV. These will quadrisect the given AB.

For because by construction, the lines LO, RS, parallel and equal, are joined by LR and OS, these also (*a*) will be parallel. In the like manner OS and QV are parallel. Therefore seeing AQ is cut into three equal parts, AI will also (*b*) be cut into so many equal parts. Likewise BC will be cut into three equal parts. Therefore the whole AB will be cut into four equal parts.

(a) *Per* 33. *l.* 1.
(b) *Per Cor. p.* 2. *l.* 6.

These two ways of practice are easier than *Euclid*'s, because fewer parallels are to be drawn.

PROP. XI. Problem.

Fig. 16. *TO find a third proportional to two right lines given (AB, BC.)*

Draw the right line AC. From BA produced, take AF, equal to BC. Through F, draw the infinite line FX, parallel to AC, which infinite, let BC produced meet in L. I say that AB is to BC, as BC to CL.

For AB : AF (*b*) : : BC : CL. But AF (*c*) is equal to BC. Therefore AB : BC : : CL; and so CL is the third proportional sought.

(b) *Per* 2. *l.* 6.
(c) *By the Construction.*

Otherwise.

Fig. 17. LET AB and BC be set at a right angle. Join AC. From C draw CX perpendicular to AC infinite; which CX, let AB produced, meet in L. I say, AB : BC : : BC : BL. It is manifest from *Corollary* 1. p. 8.

Scholium.

Scholium.

A GIVEN proportion may not only be continued in three, but alſo in infinite terms, and the whole ſum of the infinite proportional terms be exhibited. *Gregory* of *Saint Vincent* hath very handſomely proſecuted this matter, and the whole buſineſs of geometrical progreſſion in the whole ſecond book of his work. We, for the ſake of the ſtudious, will here preſent ſuccinctly the conſtruction and demonſtration of the thing propoſed.

Problem.

LET a proportion of the greater inequality be given, as A B to B C. It is required to continue this through infinite terms, and to preſent the ſum of them all. *Fig.* 19.

Let the perpendiculars A L, B O, be erected, and taken equal to the given lines A B, B C, and through L, O, let a right line be drawn, meeting with A B C produced in Z. I ſay, 1. If from C you erect the perpendicular C Q; C Q ſhall be a third proportional. Transfer Q C into C E, and from E erect E R; this ſhall be a fourth proportional. Transfer E R into E F, and erect F S; this ſhall be a fifth proportional: And ſo the proportion of A B to B C, that is, of A L to B O, will be continued through the terms A L, B O, C Q, E R, F S, *&c.* or A B, B C, C E, E F, &c. infinitely, becauſe every term (as F S) may be taken away from the remaining one F Z; for ſeeing L A (that is, A B) is leſs than A Z; F S alſo (*a*) muſt ever be leſs than F Z.

(a) *Per Coroll. Prop.* 4. *l.* 6.

I ſay, 2. A Z is equal to the whole ſum of the infinite proportionals.

Part I. ["It being ſuppoſed as before, A Z : B Z : : "A B : B C; it will be by alternating A Z : A B : : B Z : "B C. And by dividing, A Z—A B : A B : : B Z—B C : "B C; that is, B Z : A B : : C Z : B C. Therefore by "inverting A B : B Z : : B C : C Z. And by compounding "A B+B Z : B Z : : B C+C Z : C Z; that is, A Z : B Z

" : : B Z

" :: BZ : CZ."] But as AZ is to BZ, so is LA to OB; and as BZ is to CZ, so is OB to QC. Therefore also LA is to OB, as OB is to QC. In the same manner I might shew that OB is to QC, as QC to RE; and so forwards infinitely.

Part II. The whole sum of the infinite terms is neither less than AZ, nor greater; therefore it is equal. It is not greater, because seeing we have shewed above, that QC is lesser than CZ, and RE than EZ, and SF than FZ, and so on infinitely, all the terms QC, RE, SF, &c. may be infinitely set one by another in the right line AZ; so that the point Z shall never be reached. Again, the said sum will not be less, because I have above shewed AZ, BZ, CZ, to be continually proportional; and in the same manner, the same thing is shewed of the rest, EZ, FZ, &c. Seeing therefore by transfering the proportionals, QC, ER, FS, &c. into CE, EF, FI, the remainders EZ, FZ, IZ, &c. are always continually proportional, as we have already shewed; we shall at last come unto a remainder less than any given one; and therefore the sum of the proportionals shall exceed every quantity that is less than AZ; from whence itself cannot be less than AZ. Seeing therefore it is neither greater nor less than AZ, it shall be equal to it. *Q. E. D.*

Theorem.

THE difference of the first terms, the first term, and the whole sum of the infinite proportionals, are continually proportional.

Fig. 19. In the upper figure let OX be drawn parallel to AZ. Therefore LX shall be the difference of the first term AL or AB, and of the second BO, or BC. Because XO is parallel to AZ; LX shall be to XO, as (*a*) LA is to AZ. But XO is AB, and LA likewise is AB. Therefore the difference LX is to the first term AB, as AB the first term is to AZ, the whole sum. *Q. E. D.*

(a) *Per Corol.* 1. *Prop.* 4. *l.* 6.

Fig. 20. The same thing may be demonstrated universally and very briefly in every kind of quantity; thus, let there be any continual proportionals whatsoever (as well numbers, as other quantites) AZ, BZ, CZ, &c. and let them all be transferred upon the first AZ, Therefore AB, BC, CE, EF,

, F I &c. will be the differences of the proportionals; which, together with the last quantity I Z, are equal to the first A Z. Now, because if proportionals be continued infinitely, the last quantity vanisheth away, it is manifest that the differences of the infinite proportionals are equal to the first A Z. Then, because A Z is to B Z, as B Z is to C Z, and so on. By dividing, A B will be to B Z, as B C to C Z; and by converting, as A B, the first difference, is to A Z, the first quantity; so B C the second difference, is to B Z, the second quantity, and so forwards. Therefore as A B, the first difference, is to A Z the first quantity. So all the differences (that is, as I have already shewed, the first quantity A Z) are to all the quantities, that is, to the whole sum of the infinite quantities. *Q. E. D.*

PROP. XII. Problem.

THREE right lines being given (AB, BC, AF) to find a fourth proportional.

Let the two right lines be disposed, as the figure shews, and draw the right line B F, to which let the infinite right line C Z be made parallel. Let A F produced to L, meet C Z. *Fig.* 21.

I say, AB is to B C, as A F to F L, as is manifest from *Proposition* 2. of this book. Therefore F L is the fourth proportional sought.

Scholium.

OUR countryman *Bettin*, in his treasury of mathematical philosophy, doth handsomely from 35 *l.* 3. and of this, which depends not upon the present proposition, find out a fourth proportional, three being given, and a third, two being given, after this manner.

If three right lines be given, let the second C B, and the third B D, be joined right to one another, so as to make one right line, and let the first B A touch them in the point B in what angle you will. Through the points C, A, D, describe *Fig.* 22.

(a) *Per* 5. *l.* 4. describe a circle (*a*), which let AB, the first line, m... in the point Z. BZ is a fourth proportional.

(b) *Per* 35. *l.* 3. For seeing the rectangles ABZ, CBD are (*b*) equ... AB will be to BC, as BD to BZ, by the 14th of ... book, which, as was said, depends not upon this.

Fig. 23. If there be given two right lines, AB, BC; let BD, equal to BC, be joined to BC, so as to make one straight line. Then let the first AB touch BC in B in any angle. Then the rest is as before, and BZ will be the third proportional sought.

The demonstration is the same: for seeing the rectangles ABZ, CBD, are equal, AB will be to BC, as BD (that is, BC) is to BZ.

PROP. XIII. Problem.

Fig. 24. *TWO right lines given (AC, CB) to find a mean proportional.*

Let the whole compound line AB be bisected in O, and from the center O a circle be described through A and B; from C erect a perpendicular CF, meeting the circumference in F.

(c) *Per* 31. *l.* 3. I say, AC is to CF, as CF is to CB.

For let AF, BF be drawn; the triangle (*c*) AFB is right angled, and from the right angle there is drawn the perpendicular FC to the base. Therefore AC is to CF,

(d) *Per Corol.* 1. *p.* 8. *l.* 6. as (*d*) CF is to CB.

Corollary.

HENCE it is manifest, that if from any point of the circumference (as F) there be drawn a perpendicular (FC) to the diameter, this perpendicular is a mean proportional betwixt the segments of the diameter (AC, CB.)

Scholium.

Scholium.

THIS place requires, that we should say something briefly concerning the finding out of the two mean proportionals betwixt the two given lines. All the Geometricians of *Greece*, at *Plato*'s suggestion, set themselves with all their might to the solution of this problem. Divers most subtil ways of practice are recited by *Eutocius* in his commentary on *Archimedes*; as those of *Plato, Architas* the *Tarentine, Menæchmus, Eratosthenes, Philo Byzantius, Hero, Apollonius* of *Perga, Nicomedes, Diocles, Sporus, Pappus*; to whom the latter times have added *Verner, Gregory* of saint *Vincent, Renatus, Cartesius*. Out of all these we shall select three more easy than the rest.

Plato's *Method.*

IT is required to find out two means betwixt the given lines A B, B C. *Fig.* 25.

Let A B, B C be set in a right angle, and be produced infinitely towards X and Z. Then let two squares (so our *Claudius Richards* hath it) for *Plato* himself made use of one square only, but which had inserted into its side * D E, * *See* a rule moveable along D E, let two squares, I say, be *Fig.* 26. taken, and the angle D of one square be applied to the right line B X, in such certain wise, that one side may also pass through A; and to the point E, in which the other side cuts the right line B Z, let a second square be applied, which will pass through C. I say, that B D, B E, are two means betwixt the given lines A B, B C; that is, as A B is to B D, so is B D to B E and B E to B C.

The demonstration is manifest from *Corollary* 1. *Prop.* 8. *Lib.* 6. for A D E is a right-angled triangle, and from the right angle to the base there falls the perpendicular D B. Therefore by the said Corollary, as A B is to B D, so is B D to B E; and for the same cause, as B D to B E, so is B E to B C. Therefore betwixt the given right lines A B, B C, there are found two mean proportionals B D, B E. Which was the thing to be done. This manner of solving the problem is the easiest of all to be understood.

The

The Method of Philo *the* Byzantine.

Fig. 27. LET the two given right lines A B, B C, be set together at a right angle; then let the rectangle A B C D be perfected, and let D A, D C be produced infinitely, and let the diameters B D, A C be drawn, cutting each other in E. From the center E, through B, let a circle be drawn, (a) *Per* 31. *l.* 3. which because A B C is a right angle (*a*) will pass through A and C. Then let a rule be applied to the point B, so that the intercepted right lines B G, O F, may be equal. I say, that A F, G C, are two mean proportionals betwixt the given A B, B C; that is, as A B is to A F, so is A F to G C, and G C to C B.

(b) *By the Construction.* (c) *Per Corol.* 1. *p.* 36. *l.* 3. (d) *Per* 14. *l.* 6. (e) *Per Corol.* 1. *p.* 4. *l.* 6.

Demonst. Because G B, O F (*b*) are equal O G, B F, will be also equal. Therefore the rectangles O G B, B F O, that is, (*c*) the rectangles D G C, D F A, are equal. Therefore as G D is to D F so (*d*) reciprocally A F is to G C, but G D is to D F (*e*) as B A to A F. Therefore as B A is to A F, so A F is to G C. Again, because I have already shewed that A F is to G C, as B A is to A F; and since B A is to A F, as G D is to D F; that is, G C is to C B, A F will also be to G C as G C is to C B. Therefore all four B A, A F, G C, C B are, continually proportional; and therefore betwixt the given lines A B, B C, two means have been found. *Q. E. I.*

These two methods of solution, although they be ingenious and easy enough; yet because a due application of a Square and Rule is not made but by trying, they are not Geometrical.

The Method of Cartes.

Fig. 28. LET an instrument of such sort be provided, that two rules may be opened and shut about Y. Let there be inserted into these divers squares connected together betwixt themselves in the points B, C, D, E, F, G, in such sort that in the mean while that the rules Y X and Y Z are opened, the square B C may impel the square C D in the rule Y Z, and the square C D may impel the square D E in the rule Y X, and the square D E may impel F E, and

E F

EF impel or force forward FG, and so on: But so that while the rules XY and YZ are shut, all the points B, C, D, E, F, G, tend to fall upon one and the same point A. By this instrument not only two, but also four and six, yea, as many means as you will, betwixt two given right lines, may be found. Which thing can be obtained neither by the sections of a cone, nor by any Methods found out by the abovesaid authors.

For two Means, three Squares are required; for four means, five squares, and so on.

Let the lesser of the given right lines be transferred upon the rule YX, and let it be YB; the greater upon the rule YZ, and let it be YE. Let the first square be applied to the point B, and be fixed there, and let the rules be opened, until the side of the third square passeth through E. I say that YC, YD, are two means betwixt the given YB YE, that is that YB is to YC, as YC is to YD, and YD to YE.

The demonstration appears out of *Corollary* 2. *p*. 8. *l*. 6. For from the nature of the instrument, in the triangle YCD, the angle at C is a right one, and from it CB falls perpendicular upon the base YD. Therefore by the said Corollary, as YB is to YC, so is YC to YD. Again, because in the triangle YDE, the angle at D is a right one, and from it there falls the perpendicular DC upon the base YE, as YC is to YD, so is YD to YE. Therefore YB, YC, YD, YE, are four continual proportionals, betwixt the given line therefore YB, YE, there have been found two mean proportionals, QC, YD. *Q. E. I.*

If betwixt the given ones YB, YG, there be required four means, open the rules, until the side of the fifth rule FG, passeth through G. There will be YC, YD, YE, YF, four means betwixt YB, YG. The demonstration is manifest from the said corollary.

This way, although the instrument is more operose, than *Plato's*, is in very deed an excellent one; both because it doth nothing by bare trial, and because it extends itself unto four and six, and as many means as you will.

The *Deliacal* problem, to wit, the duplication of the cube, is performed by two means, and all the bodies whatsoever are encreased or diminished in a given proportion (a) by the same; like as the same thing is performed in plain

(a) *See Scho. p.* 18. *l.* 12.

(a) *Cor.* 3. *p.* 20. *l.* 6. plain figures (*a*) by one mean. *Hippocrates* first opened this way, which, as the singular and only one, all geometricians that have followed him, have embraced.

PROP. XIV. Theorem.

Fig. 29, 30. *EQual parallelograms (X, Z) which have one angle (C) equal to one (O;) have their sides also, which are about the equal angles, reciprocal; that is, (AO is to CB, as FO is to OL.)*

And if they have the sides thus reciprocal, the parallelograms are equal.

Part I. Let IL and SB, being produced, meet together in Q. The parallelogram X is to the parallelogram R, as AC is to CB (*b*); and Z is to R (*c*,) as FO to OL. But because, by the hypothesis, X and Z are equal, X is to R as Z is to R. Therefore also AC is to CB, as FO is to OL. *Q. E. D.*

(b) *Per* 1. *l.* 6.
(c) *By the same.*
(d) *By the same.*

Part II. As AC is to CB, so X is to R (*d*): And as FO is to OL, so is Z to R. But already by the hypothesis, AC is to CB, as FO to OL. Therefore X is to R, as Z is to Z. Therefore X and Z are equal. *Q. E. D.*

[Corollary. "On this depends the demonstration of the "inverse rule of proportion. For in it there is always "some rectangle given, as X; and one side of another "equal rectangle, as CB; and the other side is sought. "As therefore AC, the first side of the given rectangle "is to CB, the given side of the other rectangle; so "reciprocally, FC, the sought side, is to CL, the second "side of the given rectangle. The rectangle therefore "CB×FC, is equal to the rectangle AC×CL: And the "latter rectangle given being divided by the given side of "the former CB, the quotient will give the sought side "FC. *Q. E. I.*

PROP. XV. Theorem.

Fig. 31, 32. *EQual triangles (ACL, FCB) which have one angle (C) equal to one (O) have also their sides*

sides about the equal angles reciprocal (that is, AC is to CB, as FO to OL.)

And if they have their sides thus reciprocal, the triangles are equal.

Let the right line LB be drawn; the rest of the demonstration is the same as that of the foregoing.

Corollary.

AS well parallelograms as triangles, which have their bases and altitudes reciprocal, are equal: And so conversly.

It is manifest from the two foregoing propositions.

PROP. XVI. Theorem.

IF four right lines (AB, FI; IL, BC) be proportional, (that is, if AB be to FI as IL is to BC) the rectangle (X) under the extremes (AB, BC) is equal to the rectangle (Z) under the means, (FI, IL.) Fig. 33.

And if a rectangle under the extremes be equal to a rectangle under the means, those four right lines will be proportional.

Part I. In the rectangles X and Z, about the right, and therefore equal angles, B, I, by the hypothesis. AB is to FI, as reciprocally, IL to CB. Therefore X and Z (*a*) are equal, *Q. E. D.* (a) *Per* 14. *l.* 6.

Part II. Because X and Z are now supposed equal; therefore (*b*) about the equal angles B and I, AB is to FI, as reciprocally, IL to BC. *Q. E. D.* (b) *By the same.*

[Corollary (1.) "Hence it is easy to apply the given "rectangle Z (*c*) to the given right line AB; to wit, by "making AB : FI : : IL : BC. For BC is the rectan-"gle Z applied to the given right line AB.] (c) *Per* 12. *l.* 6.

Corollary (2.) "Upon this proposition depends the de-"monstration of the direct rule of proportion. For in it

" there is always given ſome rectangle, as C L: and ano-
" ther like rectangle is ſought, one ſide whereof is al-
" ſo given. It will therefore be, as B C, the firſt ſide of
" the rectangle given, is to E O, the ſide of the rectan-
" gle ſought; ſo directly C E, the ſecond ſide of the rect-
" angle given, is to O A, the other ſought ſide. There-
" fore the rectangle C E×O E is equal to the rectangle
" B C×O A. And the rectangle C E×E O being divided
" by B C the quotient, will give A O, the other ſide
" which was ſought. *Q. E. I.*

PROP. XVII. Theorem.

Fig. 34. *IF the three right lines (A B, F L, B C) be proportional, the rectangle under the extremes (A B, B C) ſhall be equal to the ſquare of the mean (F L.)*

And if the rectangle under the extremes be equal to the ſquare of the mean, thoſe three right lines are proportional.

Part I. Let O be taken equal to the mean F L. Becauſe therefore by the hypotheſis A B is to F L, as F L to B C, and O is equal to F L; A B will alſo be to F L as O is to B C. Therefore (*a*) the rectangle under the extremes A B, B C, is equal to the rectangle under the means F L and O, that is, is equal to the ſquare of F L.

(a) *By the foregoing.*

Part II. This is demonſtrated in like manner from the ſecond part of the foregoing.

Corollary.

Fig. 24. FROM this, taken together with the 13th, it is manifeſt, that if in a circle, F C be perpendicular to the diameter, the rectangle A C B is equal to the ſquare of F C.

(2.) If A×B be equal to the ſquare of C; then A : C :: C : B.

(3.) If A : C :: C : B; and C q be divided by A, the quotient (*b*) will be B.

(b) *Per Corol.* 2. *p.* 16. *l.* 6.

PROP. XVIII. Problem.

UPON a given right line (RS) to describe a polygon like, and in like manner posited to a given one (BQ.) Fig. 35.

Resolve the given polygon BQ into triangles. Upon the given right line RS, make the angles (*a*) R, O, equal to the angles BA. The sides then will meet together in X. Upon XS make the angles V, I, equal to the angles T, C. The sides then will meet together in Z. I say, the thing is done.

(a) *Per* 23. *l.* 1.

For because the angles RO, are equal to the angles BA, the angles E, K, must also be equal (*per* Corol. 9. *p.* 32. *l.* 1.) and because also by the construction, V is equal to T, the whole EV must be equal to the whole KT. In like manner because O, I, are equal to A, C, respectively, the whole angles OI, AC must be equal. And because V and I also are equal to T and C by the construction, Z and Q likewise must be equal (*per* Cor. 9. *p.* 32. *l.* 1.) to T and C. Therefore the polygons RZ, BQ, are mutually equiangular. It remains, that we shew that their sides also are proportional. RS is to BF, * as SX to FL; and again, SX is to FL (*b*,) as SZ to FQ. Therefore *ex æquo* RS is to SZ, as BF to FQ, &c.

* *Per* 4. *l.* 6.

(b) *By the same.*

Corollary. "Hence is derived the method of making "maps or charts, whether geographical, or chorogra-"phical, or those which surveyors of land make; and of "framing ichnographical delineations of fields, buildings, "countries: for they are nothing else but the reduction "of great figures unto like figures which are of a small "compass, which is performed by the means of this pro-"position.

PROP. XIX. Theorem.

THE proportion of like triangles (X, Z,) is duplicate of the proportion of their sides (AC, FL) which are subtended to the equal angles. Fig. 35, 37.

* Per 11. l. 6. That is, if it be made * as A C is to F I, so is F I to a third, A Q; the triangle X is to the triangle Z, as A C, the first, to the third proportional, A Q. See Definition 10. 5.

Because the triangles X Z, are like, B A will be to L I (c) as A C is to I F. But by the construction, as A C is to I F, so is I F to A Q. Therefore also B A is to L I, (d) as I F to A Q. Therefore in the triangles Q B A and Z, the sides about the angles A, I, (which, by the definition of like triangles, are equal) are reciprocal. Therefore Q B A and Z are equal (e). But the triangle X is to Q B A, as the base A C to the base A Q (f.) Therefore X is to Z, as A C to A Q. *Q. E. D.*

(c) Per 4. l. 6.
(d) Per 15 l. 6.
(e) Per 1. l. 6.
(f) Per 1. l. 6.

Corollary. "Hence is their error to be corrected, who "think that like figures are in the same proportion to one "another, that their sides are. For if of two, not only "like triangles, but also squares, pentagons, hexagons, "&c. yea, and circles also, the sides or diameters be "betwixt themselves, as 2 to 1, the figures or areas "themselves, as 4 to 1. If the sides be betwixt them-"selves, as 3 to 1, the figures themselves or areas, are "as 9 to 1; to wit, in a duplicate proportion of those "sides.

PROP. XX. Theorem.

Fig. 38. *LIKE polygons (ABCDE, FGHIK) are divided. (1.) Into like triangles (P, S, and Q, T, and R, V) in number equal. (2.) And proportional to the wholes. And (3.) The proportion of the polygon is duplicate to that of the sides, (AB, FG) which are betwixt the equal angles (B, G, and BAE, GFK.)*

Part I. Because the polygons are alike, they are mutually (*per Def.* 1. *l.* 6.) Equiangular, and their angles equal, B A E to G F K, and B to G, and B C D to G H I, and C D E to H I K, and E to K. Because therefore A B is to B C (*a*) as F G to G H, and the angles B and G are equal, the triangles P, S, (*b*) are like. In like manner it will be demonstrated, that R and V are like. Then, because the wholes, B C D, G H I, and the subducted ones, B C A, G H F,

(a) By the same.
(b) Per 6. l. 6.

GHI, are equal, the remaining ones alſo, ACD, FHI, are equal. In the ſame manner I might ſhew that ADC, FIH are equal. Therefore (*per* Corol. 9. *p.* 32. *l.* 1.) the third CAD is equal to the third HFI. Where alſo (*e*) the triangles Q and T are alike. The firſt part thereof is manifeſt. (e) *Per* 4. *l.* 6.

Part II. Becauſe P and S are alike, the proportion of P to S is duplicate to that of (*f*) CA to HF. But for the ſame cauſe alſo the proportion of Q to T is duplicate to the proportion of CA to HF. Therefore P is to S as Q to T. In the ſame manner, I will ſhew that as Q is to T, ſo R is to V. Therefore, as one antecedent, P, is to one conſequent, S, ſo all the antecedents, P, Q, R, taken together, are to all the conſequents, S, T, V, taken together; that is, ſo is polygon to polygon. Which was the other part. (f) *By the foregoing.*

Part III. The proportion of P to S is duplicate (*h*) to that of AB to FG. But the proportion of polygon to polygon is the ſame with the proportion of P to S, as I have already ſhewed. Therefore alſo the proportion of polygon to polygon, is duplicate to the proportion of AB to GF. Which was the third part. (h) *By the foregoing.*

Corollaries.

1. ALL ordinate or regular figures, as ſquares, equilateral triangles, pentagons, &c. are betwixt themſelves in the duplicate proportion of the ſides. For all regular figures are like, as is manifeſt from *Definition* 1. 6.

2. If any like figures whatſoever, the ſides AB, FG, which are placed betwixt equal angles, be known, the proportion of the figures is alſo known. As for example: Let AB be of two feet, and FG of ſix feet; and as 2 is to 6, ſo let 6 be to ſome other number; to wit 18. The leſſer figure is to the greater, as 2 is to 18, or as 1 is to 9. Now a third proportional number is found, if (*per* Corol. 3. *p.* 17. *l.* 6.) the ſecond of the given ones be multiplied by itſelf, and the product divided by the firſt. *Fig.* 38.

3. From the ſame propoſition is drawn the excellent method of encreaſing or diminiſhing any rectilineal figure in a given proportion. As if I would make a pentagon, whoſe ſide is AB, five fold of another. Find a mean proportional, BX, (*i*) betwixt the terms of the proportion given *Fig.* 39. (i) *Per* 13. *l.* 6.

(a) *Per* 18. *l.* 6. given, AB, BC; upon this frame, (*a*) a pentagon like to the given one. This shall be quintuple of the given one.

For by the 20th, the pentagon AB is to BX, which is like to it, as AB, the first, is to BC, the third proportional.

Moreover, seeing the proportion of circles also is duplicate to the proportion of their diameters, as will be shewed, *p.* 2. *l.* 12. This practice belongs likewise to circles.

Fig. 41. [Scholium. " Seeing the proportion of the squares " E, K, is duplicate of the proportion of their sides OR, " SV; from thence the duplicate proportion of the sides " OR, SV, is wont commonly to be expressed by the proportion of ORq to SVq.]

PROP. XXI. Theorem.

Fig. 40. *Figures A, (B) which are like to the same (C) are also like betwixt themselves.*

This is manifest from *Definition* 1. *Lib.* VI. and from *Axiom* 1. *Lib.* I.

PROP. XXII. Theorem.

Fig. 40, 41. *If four or more right lines (FI, LQ, and OR, SV,) be proportional; like figures, and in like sort described by them (AB and EK) must also be proportional.*

And conversly.

The demonstration of the first part is manifest. For because the proportions of A to B and E to K, are duplicate to the proportions of FI to LQ, and OR to SV, which are, by the hypothesis, equal, themselves also must be equal.

The second part is manifest also.

Fig. 24. [Corollary. " If the right line AB be cut in any manner in C; the rectangle contained under the parts AC, " CB,

" CB, is a mean proportional betwixt their squares. " Likewise the rectangle contained under the whole AB, " and one part, AC or CB, is a mean proportional be- " twixt the square of the whole, AB, and the square of " the said part, AC or CB. For (*per* Cor. 1. *p.* 8. *l.* 6.) " it is manifest, that * AC : FC : : CF : CB. There- " fore AC square : CF square : : CF square : CB square. " That is, † AC square : rectangle ACB : : rectangle " ACB : CB square. *Q. E. D.*

* *Corol.* 1. *p.* 8. *l.* 6.

† *Per* 17. *l.* 6.

" Moreover, (*per* Cor. 2. *p.* 8. *l.* 6.) BA : AF : : AF : " AC. Therefore BAq : AFq : : AFq : ACq. That " is, § BAq : BAC rectangle : : BAC rectangle : " ACq. In the same manner ABq : ABC : : ABC : " BCq. *Q. E. D.*

§ *Per* 17. *l.* 6.

PROP. XXIII. Theorem.

Fig. 42.

E*Quiangled parallelograms (X, Z) have betwixt themselves a proportion that is compounded of the proportions of their sides (AC to CB, and LC to CF.)*

That is, if you make CB to be to O, as LC to CF, X is to Z, as AC is to O.

Let IL, SB, meet together in Q. The parallelogram X (*a*) is to the parallelogram R, as AC is to CB; and R (*b*) to Z, as LC is to CF; that is, as CB is to O. Therefore *ex æquo* X is to Z, as AC is to O. *Q. E. D.*

(a) *Per* 1. *l.* 6.

(b) *By the same.*

Corollaries.

FROM hence, and from 34. *l.* 1. it is manifest.

1. That triangles which have one angle (at C) equal, have that proportion betwixt themselves, which is compounded of the proportions of the right lines AC to CB, and LC to CF. Which lines contain the equal angle.

Fig. 42.

2. That rectangles and consequently all parallelograms whatsoever, have betwixt themselves the proportion which is compounded of the proportions of the base to the base, and the height to the height. And in the same manner we reason about triangles.

3. Hence

Fig. 42. 3 Hence the proportion of triangles and parallelograms may be readily learned. Let X and Z be the parallelograms, and their bases A C, C B, and C L, C F, be their heights.
(c) *Per* 12. *l.* 6. Let it be made (*c*) as the altitude C L, is to the altitude C F, so is one of the bases C B, to O. The parallelogram X is to the parallelogram Z, as A C to O.

PROP. XXIV. Theorem.

Fig. 43. *IN every parallelogram (as S, F) the parallelograms which are about the diameter (A B) to wit, (C L, O I) are both like to the whole parallelogram, and to each other.*

By 27. 1. the angles, C, S, and L, F, are equal. By the same, E is equal to I, that is, by the same, equal to A it self; but B is common both to the whole, S, F, and the part, C L. Therefore the whole, S, F, and the part, C L, are equiangular. It remains to be shew'd, that they have the sides opposite to the equal angles proportional.

Because in the triangle B C E, B S A, C E is parallel to S A, B C (by *Corol.* 1. *p.* 4. *l.* 6.) will be to C E, as B S to S A: And C E will be to E B (by the same *Corollary*) as S A to A B. But because in the triangles E L B, A F B also, E L is parallel to A F, E B (by the same *Corollary*) will be to E L, as A B to A F. Therefore *ex æquo* C E is to E L, as S A to A F. Therefore (by *Definition* 1. *L.* VI.) C L, and the whole, C F, are like. In the same manner, I might shew O I to be like to the whole, S, F. Therefore (*per* 21. *l.* 6.) C L and O I are also like betwixt themselves. *Q. E. D.*

PROP. XXV. Problem.

Fig. 46. *TO change a given polygon (A) into another like to a given one (B.)*

Or to make a polygon equal to a given one (A) and like to another given one (B.)

Upon C F, the side of the polygon B, a like one to which is required, (by 45. *l.* 1.) make a rectangle Q, equal to B. Then upon F I (by the same *Proposition*) make a rectangle

rectangle R equal to A. It is manifest, that C F and F I do make one right line. Betwixt C F and F I find a mean proportional F L (*a.*) Upon this, (*per* 18. *l.* 6.) make a polygon like to the given one B, this must also be equal to the given one A. (a) *Per* 13. *l.* 6.

For seeing by the construction, C F, F L, F I, are three proportionals, the polygon B is to the polygon like to it, which is made upon FL, as C F is to F I (*per* 20. *l.* 6.) and *Definition* 10. *l* 5. that is, (*per* 1. *l.* 6.) as Q is to R. Therefore also by changing, as the polygon B is to Q, so is the polygon F L to R. But by the construction, the polygon B is equal to Q Therefore also the polygon upon FL, which is like to B, is equal to R; that is by the construction, to the given A. That therefore is done which was required.

PROP. XXVI. Theorem.

LIKE *parallelograms* (*B D, F N,*) *having a common angle* (*A*) *are about the same diameter.* Fig. 44.

Draw the right lines A E, C E. If you deny that A E C is a common diameter to the parallelograms B D and F N; let another right line A G C, which cuts F E in G, be the diameter of B D, and draw the parallel G H. The parallelograms F H, B D, will be therefore about the common diameter A G C, and consequently (by 24. *l.* 6.) will be like. Therefore, (*per Definition* 1. *l.* 6.) will be like. Therefore, as B A to A D, so is F A to A H. But also, as B A to A D, so is F A to A N, seeing B D, F N, are like by the hypothesis. Therefore F A is to A H, as the same F A is to A N. Which is absurd.

PROP. XXVII, XXVIII. XXIX.

THESE *cause trouble to, and perplex beginners, and are scarce of any use.*

PROP.

PROP. XXX. Problem.

Fig. 45. *TO cut a given right line (AB) ſo that the whole (AB) ſhall be to one ſegment (AC) as the ſame ſegment is to the remainder (CB.)*

That is, as geometricians ſpeak, to cut a line in extreme and mean proportion.

By 11. *L.* 2. ſo cut AB in C, that the rectangle under AB, CB, may be equal to the ſquare of AC. I ſay the thing is done.

For by the 17th of this book, as AB is to AC, ſo is AC to CB.

The force of this ſection of a line is admirable in the inſcribing and comparing regular bodies.

PROP. XXXI. Theorem.

Fig. 47. *IF from the ſides of a rectangular triangle (ACB) like figures whatſoever be deſcribed, that which is oppoſed to the right angle, will be equal to the two others (L, R) taken together.*

Here *Propoſition* 47. *l.* 1. is made univerſal.

From the right angle C, let the perpendicular CO be let down. Becauſe (*per* Corollary 2. *P.* 8. *L.* 6.) AB, BC BO, are three proportionals, F ſhall be to the figure R, which is like to it, as AB the firſt, to BO the third proportional, (to wit, by 20. *l.* 6. and *Definition* 10. *L.* 5.) Again, becauſe (by the aforeſaid Corollary) BA, AC, AO, are three proportionals, the figure F ſhall (by the aforeſaid *Propoſition* and *Definition*) be to L, which is like to it, as BA the firſt, to AO the third proportional. Becauſe therefore F is to R, as AB is to BO; and the ſame F is to L, as AB to AO; F ſhall alſo be to R and L taken together, as AB is to BO, AO, taken together. But AB is equal to the two, BO, AO. Therefore alſo F ſhall be equal to the two, R and L. *Q. E. D.*

Corollary,

Corollary.

FROM this propoſition we can eaſily find one rectilinear figure, equal and like to any number of rectilinear figures whatſoever, by the ſame method, whereby, *Prop.* 1. *Schol. p.* 47. *l.* 1. one ſquare is found equal to any number of given ſquares whatſoever. Only in the demonſtration, let 31. *l.* 6. be cited inſtead of 47. *l.* 1.

Corollary (2.) " A circle upon the hypothenuſe of a " rectangle triangle, is equal to two circles deſcribed " upon the ſides, for all circles are like amongſt them- " ſelves; and are to one another as the ſquares of their " diameters, by the ſecond of the twelfth book.

Corollary (3.) " From hence we may derive that qua- " drature of lunets (or little moons) which *Hippocrates* of " *Chios* firſt taught. *Fig.* 54.

" For let A B C be a rectangled triangle; and B A C a " ſemi-circle to the diameter B C : B N A a ſemi-circle " deſcribed on the diameter A B; A M C a ſemi-circle " deſcribed upon the diameter A C. Thus therefore the " ſemi-circle B A C is equal to the ſemi-circles B N A and " A M C together. If therefore you take away the two " ſpaces B A, A C, common to both ſides, there will be " left the two lunets B N A, A M C, bounded on both " ſides with circular lines equal to the rectilineal triangle " B A C. And if the line B A be equal to the line " A C, and you let fall a perpendicular unto the hypothe- " nuſe B C, the triangle B A O will be equal to the lunet " B N A, and the triangle C O A equal to the lunet C M A. " *Q. E. I.*

PROP. XXXII.

THIS is hardly of any uſe, and hath nothing remarkable in it.

PROP. XXXIII. Theorem.

Fig. 48. *IN the ſame or equal circles, the angles, whether at the centers (as A B C, F O D) or at the circumference (as A R C, F S D) have that proportion betwixt themſelves, which the arches (A K C, F G D) on which they ſtand, have. Underſtand the ſame thing of ſectors.*

As for the angles at the center, and the ſectors, it will be demonſtrated altogether in the ſame manner in which, *Prop.* 1. of this book, it was demonſtrated, that triangles of the ſame height are as their baſes: Only where, *Prop.* 38. *l.* 1. is cited there, let *Prop.* 29. *l.* 3. be cited here.

And becauſe the angles R and S, at the circumference, are halves of the angles ABC, FOD, at the center, that which hath been demonſtrated of theſe will be manifeſt alſo of thoſe.

Corollary.

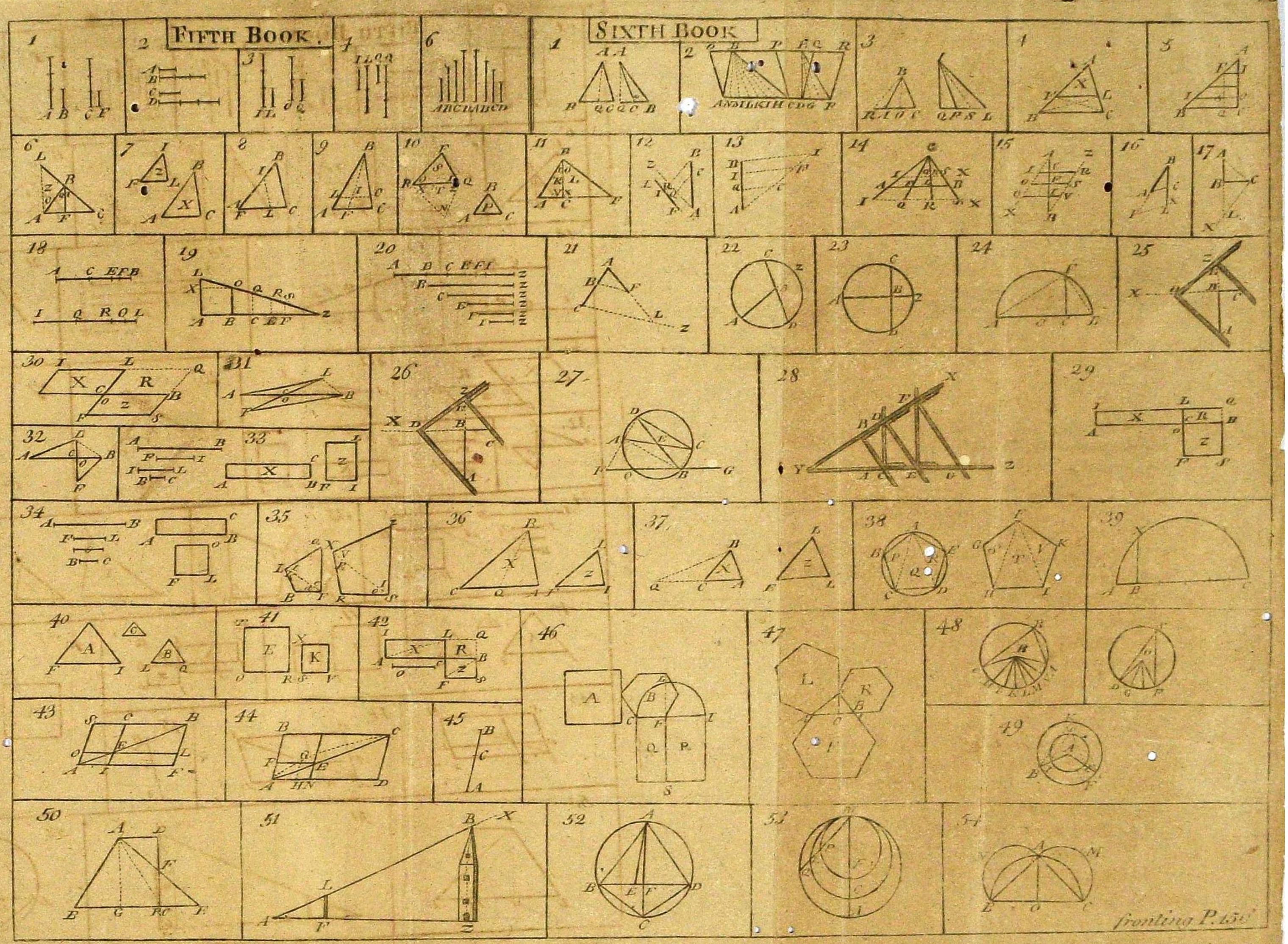
Fig. 49. 1. THE angle (B A C) at the center, is to four right angles, as the arch B C on which it ſtands, is to the whole circumference.

For as B A C is to the right angle B A F, ſo by this, 33, the arch B C is to the quadrant B F. Therefore the angle B A C is to four right angles, as the arch B C is to four quadrants, that is, the whole circumference.

The arches I L, B C, of unequal circles, which do ſubtend equal angles, whether at the center, as I A L and B A C, or at the circumference, are like arches.

For the arch I L is (by *Corollary* 1.) to its circumference, as the angle I A L, that is, B A C is to four right angles; and the arch B C is to its circumference (by the ſame *Corollary*) as the ſame angle B A C is to four right ones. Therefore I L is to its circumference, as B C is to its. Therefore (by *Defin.* 4. *l.* 6.) the arches I L and B C are like.

3. The


FIFTH BOOK
SIXTH BOOK
fronting P.156

3. The ſemi-diameters (A B, AC) do take away from concentrical circumferences like arches, I L, B C. This is manifeſt from *Corollary* 2.

4. The ſegments (B K C, I O L) which contain equal angles (K, O) are like.

For by *Corollary* 2. the Arches B C, I L, and conſequently the Angles B K C, I O L, are like.

THE

THE ELEMENTS OF EUCLID.

BOOK XI.

With Us the Seventh.

TO the six first books *Euclid* subjoins the elements of numbers, comprehended in the three following, the seventh, eighth and ninth, to which he also adjoins a tenth, concerning incommensurable quantities. We pass immediately from planes to solids; purposing to treat of numbers separately: Seeing it will, I suppose, be more commodious for learners, if the elements of geometry be not interrupted, by treating of any other matter, but be had altogether. Nevertheless, when we shall cite the propositions of this and the following book, we shall not call these books the seventh, and the eighth, but the eleventh and the twelfth, lest if we should depart from the every where received order of *Euclid*, the citation of propositions should thereby be rendered more intricate.

This book in a sort contains two parts: In the first, are laid the foundation on which the whole doctrine of the solids, that is, of bodies, depends. In the other, the affections of parallelopipeds are propounded.

" This eleventh book of elements sets forth the first " principles of solids. Nor can indeed the properties of " bodies be known without it; and if we set upon almost " any part of the mathematicks, without the knowledge " of solids, we shall labour in vain, or be at least at a great " loss,

" loss. For the spherical doctrine of *Theodosius*, spheri-
" cal trigonometry also, a great part of practical geome-
" try, statics and geography, depends upon it; and what
" things occur of any great difficulty in the art of dial-
" ling, in the conic sections, astronomy, dioptrics or
" optics, do all become more easy, the principles of solids
" being once understood. So that those who have deli-
" vered the elements of geometry, leaving out and setting
" aside this and the following book, are to be reckoned
" to have delivered the same very imperfectly.

DEFINITIONS.

1. A solid, or body, is that which hath length, breadth and thickness.

2. The extreme of a solid is a surface. *Fig.* 1 *l.* 11.

3. The right line (A B) is to the plane (C C) right or perpendicular, when it makes right angles (B A C, B A C) with all the right lines (C A) in the plane (C C) by which it is touched.

4. A plane is right or perpendicular to a plane, when all the right lines (L Q) which are drawn in one of the planes perpendicular to the common section (X R) are right or perpendicular to the other plane (A B C O.) *Fig.* 2.

5. If the right line (O L) stands upon a plane not at right angles, and from its highest point (L) there be drawn to the plane the perpendicular (L P) and (O P) be joined; the angle (L O P) is said to be the inclination of the line (O L) to the plane. *Fig.* 3.

6. If the plane (R E) doth not stand perpendicularly upon the plane (L Q) the inclination of one to the other is the acute angle (A B C) which is contained by the right lines (A B and B C) which are drawn in both planes perpendicular to the common section (O E.) *Fig.* 4.

7. A plane is said to be alike inclined to a plane, as is some other plane to another, when the said angles of their inclinations are equal.

8. Parallel planes, are those which being continued every way, are always distant from each other by equal intervals.

9. Like solid rectilinear figures are those which are contained under like planes, in number equal.

10. A solid right-lined angle, is that which is contained under plain angles more than two (B A C, C A C, O A B) which *Fig.* 5.

which are not in the ſame plane, meeting together in one point.

11. Equal ſolid angles are thoſe, which being conceived to be put each within the other, do agree or perfectly co-incide.

Like as a plain angle is a mutual inclination of lines, ſo a ſolid angle is an inclination of ſurfaces. Concerning both therefore we muſt reaſon in the ſame manner.

Fig. 6,7,8. 12. A priſm is a ſolid figure, comprehended by planes, amongſt which two oppoſite ones [O F E, A C B] are parallel, equal and like.

Fig. 8. 13. A parallelopiped is a ſolid, contained under ſix quadrilateral planes, of which the oppoſites are parallel.

14. If ſix planes, in which the oppoſites are parallel, be ſquares, the ſolid contained by them will be a cube.

PROPOSITION I. Theorem.

Fig. 9. *ONE part (AC) of a right line cannot be in a plane (OE;) and another part (CB) out of it.*

It is clear of itſelf, from the definition of a plane and a right line. See *Defin.* 4. and 7. *L.* 1.

PROP. II. Theorem.

Fig. 10. *EVERY triangle is in one plane: And two right lines cutting each other, are in the ſame plane.*

For if a plane be applied to one of its ſides, and to the point of meeting the other two, it will be evident that the whole triangle is in that plane.

PROP. III. Theorem.

Fig. 11. *IF two planes (AB, CD) cut each other (EF) their common ſection is a right line.*

It

It is manifest from the definition of a plane.

But we may demonstrate it thus. If E F, the common section, be not a right line, let there be drawn in the plane C D the right line E O F, and in the plane A B, the right line E Q F. The two right lines therefore, E O F, E Q F, will inclose a space. Which is absurd.

PROP. IV. Theorem.

IF a right line (B A) be perpendicular to two right lines (C A X, F A S) which cut each other; it will also be perpendicular to the plane which is drawn through them. *Fig.* 12.

If you deny it, let another right line, B Q, be perpendicular to the plane of the right lines A C, A F. Join A Q, and to this in the plane F A C, draw the perpendicular Q O. This being produced, will necessarily cut (as is gathered from *Schol. Prop.* 31. *l.* 1) one of the right lines C A X, F A S or both, wheresoever the point Q shall be. Therefore let it cut C A X in O, and let B O be joined. Because therefore the angle B A O is, by the hypothesis, a right one;

The square of B O shall be equal to B A squ. + A O squ. } (*b.*) (b) *Per* 47. *l.* 1.

But because B Q is supposed perpendicular to the plane F A C, and consequently (by *Definition* 3. *l.* 11.) makes a right angle B Q A with A Q:

B A squ. is equal to B Q squ. + A Q squ. } (*d.*) (d) *Per* 47. *l.* 1.

And because the angle A Q O is, by the construction, a right one;

A O squ. is equal to O Q squ. + A Q squ. } (*e.*) (e) *By the same.*

Therefore B O squ. is equal to + B Q squ. + O Q squ. + A Q squ. twice taken.

L There-

Therefore ſquare B O is greater than the ſquares of B Q and OQ; and (as is clear from *Prop.* 47. *l.* 1) conſequently B Q O is not a right angle. Therefore B Q is not perpendicular to the plane (by *Definition* 3. *l.* 11) C A F. Therefore the propoſition is manifeſt.

Scholium.

FROM its being ſuppoſed that B Q is perpendicular to the plane F A C; it is directly demonſtrated that B Q is not perpendicular to that plane; and conſequently from the denial of the aſſertion of the theorem, the ſame aſſertion is directly proved. This demonſtration, as to the ſubſtance of it is *John Cierman*'s.

PROP. V. Theorem.

Fig. 13. *IF three right lines (B A, C A, F A) be perpendicular to the ſame right line (A R) at the ſame point (A;) thoſe three will be in one plane.*

For, if it may be, let one of them B A be in another plane (R O) which may cut L Q, the plane of the other two, C A, F A, in the right line A O. Becauſe, by the hypotheſis, R A ſtands perpendicularly upon the two, C A, F A, it will be perpendicular to the plane L Q by the foregoing.) Therefore R A makes a right angle with A O (by *Definition* 3. *l.* 11.) But alſo, by the hypotheſis, R A B is a right angle. Therefore the angles R A B and R A O are equal. Which is abſurd.

PROP. VI. Theorem.

Fig. 14. *RIGHT lines (A B, C D) which are perpendicular to the ſame plane (C F) are parallel.*

It might be taken for granted, as a thing of itſelf known; but we may demonſtrate it thus:

BD

BD being joined, make in the plane FE the line DG perpendicular to BD, and equal to BA; and let DA, GA, GB, be joined. The right lines BD, DG, are equal to BD (*a*) and BA; and the angles BDG, (*b*) DBA are right ones. Therefore (*per* 4. *l.* 1.) AD, BG, are equal. Therefore the triangles ABG, GDA, are equilateral to each other, and consequently the angles ABG, ADG, are equal. But ABG *by Defin.* 3. *l.* 11. is a right angle. Wherefore ADG is also a right one. But BDG also, by the construction, and CDG, by *Defin.* 3. are right angles. Therefore GD is perpendicular to the three lines CD, AD, BD. Therefore CD is (*c*) in one plane with AD and BD. But AB also is in plane (*per* 2. *l.* 11.) with AD and BD. Therefore AB, CD are in one plane. Therefore seeing the angles ABD, CDB (by *Defin.* 3. *l.* 11.) are right ones, AB, CD, will (*per* 29. *l.* 1. and *Defin.* 36. *l.* 1.) be parallel lines. *Q. E. D.*

(a) *By the Construction.*
(b) *Per Def.* 3. *l.* 11.
(c) *By the same.*

PROP. VII. Theorem.

A Right line (EF) cutting the right lines (AB, CD) placed in the same plane, is in one and the same plane with them. Fig. 15.

It might be taken for granted. But he that will may thus demonstrate it.

Let another plane cut the plane of the right lines AB, CD, in the points EF. If now EF is not in the plane of AB, CD; EF will not be the common section. Let EGF therefore be so. Therefore (*per* 3. *l.* 11.) EGF is a right line; the two right lines therefore EF, EGF, inclose a space. Which is absurd.

Corollary.

HENCE it follows that if EF cut the parallels AB, CD, it is in the same plane with them. For (by *Definition* 36. *l.* 1) any two parallels are in the same plane.

PROP. VIII. Theorem.

Fig. 14. *IF two parallels (AB, CD) one (AB) be perpendicular to a plane EF, the other (CD) will be perpendicular to the same plane.*

It might be taken for granted. If the demonstration be required, it is as follows:

" BD, AD, being drawn; in the plane EF, make " GD perpendicular to BD. It will also (see the demon" stration of *Prop.* 6. *l.* 11.) be perpendicular to AD. " Therefore (*per* 4. *l.* 11.) GD will be perpendicular to " the plane ABD, that is, (by the foregoing corollary) " to the plane CBDA. Wherefore (*per Defin.* 3. *l.* 11.) " CDG is a right angle. But the angle CDB is also a " right one; forasmuch as with ABD, which (*per Defin.* " 3. *l.* 11.) is a right angle, it maketh two right ones " (*per* 27. *l.* 1.) Therefore (*per* 4. *l.* 11.) CD is perpen" dicular to the plane GDB or EF. *Q. E. D.*

PROP. IX. Theorem.

Fig. 16. *RIGHT lines (AB, EF) which are parallel to the same right line (CD) although they be not in the same plane with it, are also parallel betwixt themselves.*

Although it might be taken for granted, yet we will demonstrate it thus:

In the plane of the parallels AB, CD, draw GK perpendicular to CD. Likewise in the plane of the parallels EF, CD, draw HK perpendicular to CD. Therefore (a) CK is perpendicular to the plane GKH. Therefore, seeing AG, EH, be parallel to CK, the same AG, EH (b) will be perpendicular to the plane GKH. Therefore AG, EH (c) are parallel. *Q. E. D.*

(a) *Per* 4. *l.* 11.
(b) *Per* 8. *l.* 11.
(c) *Per* 6. *l.* 11.

PROP.

PROP. X. Theorem.

IF two right lines (AC, BC) be parallel to two right ones (DF, EF;) albeit they be not in the same plane, they comprehend equal angles (C and F.) Fig. 17.

Let CA, CB, be made equal to FD, EF, and let DE, AB, DA, FC, EB, be drawn. Seeing AC, FD, are parallel and equal, AD also and CF will (*a*) be parallel and equal. In like manner I might shew BE, CF, to be parallel and equal. Therefore AD, BE, are also parallel (*b*) and equal, *per Axiom* 1. Therefore, *per* 33. *l.* 1. AB, DE, are equal. Seeing therefore the triangles BAC, EDF, are equilateral to each other, the angles C and F (*c*) are equal. *Q. E. D.*

(a) *Per* 33. *l.* 1.
(b) *By the foregoing.*
(c) *Per* 8. *l.* 1.

PROP. XI. Problem.

TO draw a perpendicular to a given plane (AB) from a point given without it (C.) Fig. 18.

The construction. In the plane AB, draw any right line, as DF, unto which, from C, erect the perpendicular CE. Then in the plane AB, through E, draw AEM perpendicular to the same DF. Then to AM, from C, draw the perpendicular CG. I say, that CG is perpendicular to the plane AB.

Through G let HG be drawn parallel to DF. By the construction, DE is perpendicular to CE and EM. Therefore DE is perpendicular to the plane CEM (*d*,) as also is HG (*e*) Therefore, by *Defin.* 3. *l.* 11. CG is perpendicular to HG. But CG, by the Construction, is also perpendicular to EM. Therefore (*f*) CG is perpendicular to the plane AB. Which was the thing proposed.

(d) *Per* 4. *l.* 11.
(e) *Per* 8. *l.* 11.
(f) *Per* 4. *l.* 11.

Scholium. "In practice thus. Let there be a cord "or rule fastned to the given point A; And from the "same let there be described by the end of it B in the "plane given, the circle BCFL. The line AK, which "connects the given point and the center of the circle, "will be perpendicular to the given plane. Fig. 20. *l.* 12.

PROP. XII. Problem.

Fig. 19. *FROM a given point (A) in any plane (E F) to erect a line perpendicular to the same plane.*

From any point D, without the plane E F, make D B (by the foregoing) perpendicular to the plane E F. And B A being joined, draw A C parallel to D B. I say the thing is done. The demonstration is manifest from *Prop.* 8.

Corollary.

IN practice, from the given point, a perpendicular is erected to the given plane, if a square O K N be applied to the given point [and be turned round.]

PROP. XIII. Theorem.

Fig. 20. *LINES drawn from the same point cannot be both perpendicular to the same plane (A B.)*

For if they were, they would, (by *Prop.* 6) be parallel. Which cannot be.

PROP. XIV. Theorem.

Fig. 21. *IF the same right line (A B) be perpendicular to two planes (F G, L Q.) the planes will be parallel.*

Let there be taken in either of the planes, as F G, any point C, from which let C E be drawn parallel to A B, and meeting the plane L Q in E. Then C E (*per* 8. *l.* 11.) will be perpendicular to both planes F G, L Q. Wherefore if A C, B E be joined, the angles A B, (by *Def.* 3. *l.* 11.) will be right ones. Therefore (*per* 29, *l.* 1.) A C, B E, are parallel. Therefore A C E B is a parallelogram; and consequently C E, which hath been already shewn to be

be perpendicular to both planes, is equal (*per* 34. *l.* 1.) to AE. In the same manner I might shew that all the perpendiculars to planes are equal. Therefore (by *Defin.* 8. *l.* 11.) the planes are parallel. *Q. E. D.*

PROP. XV. Theorem.

If two right lines (B A, C A) touching each other be parallel to two right lines which also touch one another (E D, F D;) the planes likewise which are drawn through them will be parallel. Fig. 22.

From A, let there be drawn A G, perpendicular to the plane E F, and let G H, G I, be parallel to D E, D F. These (*per* 9. *l.* 11) will also be parallel to A C, A B. Seeing therefore the angles I G A, H G A, be, by *Def.* 3. *l.* 11. right; C A G, B A G, will also (*a*) be right angles. (a) *Per* 27. *l.* 1. Therefore, G A, which is perpendicular to the plane E F, will also be perpendicular to the plane B C (*b.*) Therefore (b) *Per* 4. *l.* 11. the planes B C, E F, are, by the foregoing, parallel. *Q. E. D.*

PROP. XVI. Theorem.

A Plane (E H F G,) cutting parallel planes (A B, C D,) makes the sections in them (E H, G F) parallel. Fig. 23.

If not, seeing they be in the same intersecting plane, they will meet somewhere, by *Schol. Prop.* 21. *l.* 1. as in I. Wherefore seeing the whole lines H E I, F G I, be in the planes * A B, C D, produced, these planes also will meet in * *Per* 1. *l.* 11. I. Which is absurd, and contrary to *Defin.* 8. *l.* 11.

PROP.

PROP. XVII. Theorem.

Fig. 24, *PArallel planes c[illegible]t right lines ([illegible]D and G H) proportionally.*

Let the right lines B H, G D, be [illegible] the p[illegible]nes P V, T Q; and likewise let B G be drawn meeting [illegible] plane R S in F, and let F C, F I be joined. The plane of the triangle B G D cutting parallel planes, makes the sections C F, D G parallel, by the foregoing. Therefore B C is to C D, as B F (*e*) to F G. Again, the triang[illegible] B H G cutting parallel planes, make the sections, by t[illegible] going, B H, F I, parallel. Therefore H I is to I G, as B F to F G; that is, as I have already shewed, [illegible] B C is to C D. *Q. E. D.*

(e) *Per* 2. *l* 6.
* *Per* 2. *l.* 6.

PROP. XVIII. Theorem.

Fig. 25. *IF a right line (F E) be perpendicular to a plane (A B,) all the planes which are dr[illegible]wn through it are perpendicular to the same plane (A B.)*

Let the plane G C be drawn through F E, making C D the common section with A B; and let the lines H K be drawn in the plane G C, perpendicular to the common section C D. Now seeing, by the construction, H K is perp[illegible]dicular to the same common section to which F E is perpend[illegible]ular, by the hypothesis, K H and F E must be parallel, by [illegible] *l.* 1. Therefore H K is also perpendicular to the plane A B, *per* [illegible] Therefore the plane G C, is perpendicular to the plane A B, *p*[illegible] *Definition* 4. *l.* 11.

PROP. XIX. Theorem.

Fig. 26. *IF two planes (M F, G D) cutting e[illegible]ch other, be both perpendicular to the same plane (A B;) their common section also will be perpendic[illegible]lar to that plane (A B.)*

F[illegible]r

F[illegible] seeing, by the hypothesis, the plane MF is per[illegible]lar to the plane AB; it is manifest, by *Definition* 4[illegible] there may be drawn in the plane MF, from the [illegible]ular to the plane AB; namely, that [illegible] plane MF, is perpendicular to the [illegible]mon [illegible] Again, by the hypothesis, GD is [illegible]pendicular [illegible] plane AB; 'tis evident, in the plane [illegible]D, may be drawn from the point L, a perpendicular to the plane AB. But from the point L (*a*) there can be erected only one perpendicular to the same plane AB. Therefore the perpendicular to the plane AB, which is drawn from the point L, must be found in both the planes, M[illegible]D, and consequently LK, the common section o[illegible] two planes, MF and GD, is perpendicular to the plane, A[illegible] *Q. E. D.*

(a). *Per* 13 *l.* 11.

PROP. XX. Theorem.

If a solid angle (A) is contained under three plain angles (BAC, CAD, DAB;) any two of these is greater than the third. Fig. 27.

If the three angles be equal, the assertion is manifest at first sight; and it is as certain, if they be unequal. For let BAD be the greatest; and from BAD, cut off BAE, equal to BAC, and make the line AC equal to AE. And through E, let there be drawn a right line meeting AB and AD, in B and D, and let BC, DC, be joined. because, by the construction the angles BAE, BAC, are equal, as likewise the sides BA, AE, equal to the sides BA, AC, the bases also BE, BC, will be equal (*b*.) And because BC. CD, (*c*) are greater than BD, the equal, BE, BC, being taken away, there remains CD greater than ED. But the sides EA, AD, are, by the construction, equal to the sides, CA, AD. Therefore the angle (*d*) CAD is greater than the angle, EAD. Seeing therefore the angle BAC is equal, by the construction, to the angle BAE, those two angles together, BAC, CAD, are greater than the whole, BAD. *Q. E. D.*

(b) *Per* 4. *l.* 1.
(c) *Per* 20. *l.* 1.
(d) *Per* 25. *l.* 1.

PROP.

PROP. XXI. Theorem.

THE plain angles constituti[ng any solid angl]e whatsoever, are less than [four righ]t ones.

Fig. 28. Let A be a solid angle; let the right lines, BC, C[D], DE, EF, FB, be subtended to the plain angles which make up the solid one, so that those right lines be all in one plane. Which being done, there is constituted a pyramid, whose base is the polygon BCDEF; A is the top, and it is contained under so many triangles, G, H, I, K, L, as there are plain angles which compose the solid angle A. And now, because the two angles ABF, ABC, are, by the foregoing, greater than the third, FBC; and the two, ACB, ACD, are greater than the third, BCD, and so on: All the angles of the triangles G, H, I, K, L, about the base, as taken together, are greater than all the angles of the base, B, C, D, E, F, taken together. But the angles of the base, together with four right ones, make twice so many right angles, by *Theorem* 2. *Schol.* after 32. *L.* 1. as there are sides, or which is the same, as there are triangles. Therefore all the angles of the triangles about the base, together with four right ones, make more than twice so many right angles as there are triangles. But the same angles about the base, together with the angles that compose the solid make up (*a*) twice so many right angles as are the triangles. It is manifest therefore, that the angles which compose the solid angle A, are less than four right ones. *Q. E. D.*

(a) P. 32. *l.* 1.

Corollary.

FROM this and the foregoing, it is obvious to collect, that a solid angle may be composed of any three plain angles, which are less than four right ones, if so be that any two of them be greater than the other.

Scholium.

Scholium.

FROM this propofition is demonftrated that famous theorem, that only three regular and equal plain [illegible]s can contain a body; to wit, equilateral triangles, either 4, or 8, or 20; 6 fquares, and twelve pentagons. [illegible] confequently, that there are only five regular bodies. A *Pyramid*, which is contain'd under 4; an *Octaedrum*, which is comprehended by 8; and an *Icofiedrum*, which is bounded by 20 equilateral triangles; a *Cube*, which is contain'd under 6 fquares; and the *Dodecaedrum*, under 12 regular and equal pentagons. Now a body is called regular, which is comprehended under regular and equal planes.

Demonft. A folid angle cannot be compos'd of only two equilateral triangles; three, at leaft, are requir'd.

Of three equilateral triangles meeting in one point, there may be conftituted the folid angle of a *Pyramid*, of four, the folid angle of an *Octaedrum*; of five, the folid angle of an *Icofaedrum*: Forafmuch, as both 3, 4. and 5 angles of an equilateral triangle are lefs than 4 right ones, as is gathered from *Corollary* 12. *Propofition* 32. *L.* 1.

And becaufe three angles of a regular pentagon, as is gathered from *Corollary*, *Prop.* 11. *l.* 4. are lefs than four right ones, three pentagons, meeting in one point, will conftitute a folid angle, that of the *Dodecaedrum.*

That of the three fquares, meeting in one point, may be compos'd the folid angle of a *Cube*, is manifeft of itfelf. And thus there arife five regular bodies.

But that there are no more than thefe five, is thus proved.

Six angles of an equilateral triangle make juft four right ones. For one is two thirds of one right one; and therefore fix fuch will make, by *Corol.* 12. *Prop.* 32. *l.* 1. twelve thirds of one right one, that is, four right ones. And therefore of fix equilateral triangles a folid angle cannot be compos'd, much lefs of more.

That of four fquares a folid angle cannot be made, much lefs of more, is manifeft in itfelf.

Four angles of a regular pentagon are greater than four right ones. For, by *Coroll. Prop.* 11. *l.* 4. each of them make

make six fifths of one right one. Therefore a solid angle cannot be made of four such pentagons; much less of more.

Nor can a solid angle be compos'd of any other regular figures whatsoever. Three angles of a regular hexagon, by *Corollary* 2. *Prop*. 15. *l*. 4. are equal to four right ones. For one makes four thirds of one right one; and therefore three make twelve thirds of one, that is, four entire right ones. Therefore of three hexagons a solid angle cannot be made up; much less of more.

But seeing three angles of a regular hexagon are equal to four right ones, three angles of any other figure whatever greater than an hexagon, as of an heptagon, octagon, &c. will be greater than four right ones. Wherefore it is manifest, that the rest of the regular figures are all incapable of composing a solid angle and consequently, that there can be no regular bodies beside the five foregoing.

PROP. XXII. XXIII.

ARE very prolix, and tedious to beginners, and scarce at any time come into use.

PROP. XXIV. Theorem.

Fig. 29. *THE planes which contain a parallelopiped are* (1.) *Parallelograms.* (2.) *The opposite ones are like.* (3.) *The planes are equal.*

(a) *Per* 16. *l*. 11. Part I. The plane AF, cutting the planes BD, FH, which by *Defin*. are parallel, makes (*a*) the sections BA, FE, parallel. Again, the plane AF, cutting the planes AH, BG, which, by the same definition, are parallel, (by the same) makes the section AE, BF, parallel. Therefore BAEF is a parallelogram. By the like argument the rest of the parallelopiped may be prov'd to be parallelograms.

(b) *Per* 10. *l*. 11. Part II. Because it is manifest from the first part, that AB, BC, are parallel to EF, FG; the angles ABC, EFG, must be (*b*) equal. Wherefore seeing the alternate

sides

sides also are equal, the opposite parallelograms B D, F H, are like or similar. And the same of the rest.

Part III. This is manifest from the first part, and fourth, or eighth of the first book.

PROP. XXV. Theorem.

IF a parallelopipedon (G F D I) or any prism whatever be cut by a plane (N P) that is parallel to the opposite sides; there will be this proportion, as the base (D C P O) is to the base (O P F I) so is the solid (G P) to the solid (N F.) Fig. 30.

This is demonstrated in the same manner as *p.* 1. *l.* 6.

Corollary.

A Prism cut by a plane parallel to the opposite planes, hath a section like and equal to the opposite planes.

PROP. XXVI, XXVII.

A*RE not necessary.*

PROP. XXVIII. Theorem.

A Plane passing through the diameters of opposite planes (A C, E G) cuts the parallelopiped into two equal prisms. Fig. 29.

Because (*a*) B G, H E, are parallelograms; C G, A E, are equi-distant from the same B F. Therefore (*b*) they are also parallel betwixt themselves, and consequently are in one plane. Therefore the right lines A C, E G, are (*c*) in one plane. But now that a plane drawn through them doth cut the parallelopiped into two equal prisms, is thus shew'd. Let the prism A E G C D H be understood to be so constituted upon its plane A E C G, that the angles D, H, bend

(a) Per 24. *l.* 11.
(b) Per 9. *l.* 11.
(c) Per 7. *l.* 11.

bend towards the angles B, F. It is manifest, that it will yet be betwixt the parallel planes B A D C, F E H G. But then D must needs fall upon B, and H upon F. For let D fall without B, if it may be, and in N. The angle (d) Per 27. l. 1. B A C (*d*) is equal to the angle D C A. But D C A is equal to N A C, for it is one and the same angle. Therefore B A C and N A C are equal: Which is absurd. Therefore D falls upon B; and for the same cause, H upon F. Therefore the prism A E G C D H coincides with the prism A C G E F B, and consequently they are equal, by *Axiom* 7.

PROP. XXIX, XXX. Theorems.

Fig. 31. *THE parallelopipeds (F E A G K I M C) and (F E B H L O M I) which have the same base (E F I M) and the same altitude, and consequently exist between parallel planes (E F I M) and (G A O L) are equal.*

For they either exist betwixt the lateral parallel planes E A O M and F G L I, or not. Let the first be suppos'd, from the 24th of this, and the 8th of the first book, it is manifest, that the triangles A E B, C M O; likewise G F H, K I L, are equilateral and equiangular, to each other. Wherefore, as in the foregoing, I might shew that the prisms C M O L I K, and A E B H F G, being laid upon each other will coincide, and consequently, by *Axiom* 7. are equal. Wherefore the common solid F E B H K C M I being added, the whole parallelopipeds F E A G K I M C and F E B H L O M I are equal. *Q. E. D.*

Then let the parallelopiped F X Q E M I P R not exist betwixt the same lateral parallel planes with the parallelopiped F E A G K C M I. Here, because, by the hypothesis, G K, A C, R P, Q X, are in one plane, which is parallel to the base E F I M; let R P, Q X, cut G K in L and H, and A C in O and B; and let E B, M O, F H, I L, be join'd. It is easy now to shew, that the planes containing the solid F E B H L O M I, are parallelograms, the opposite ones of which are equi-distant, and consequently that the solid is, by *Defin.* 13. *l.* 11. a parallelopiped. But to this, by the first part, the parallelopipeds F X Q E M I P R,

and

and F E A C K S M I, are each of them equal. Therefore they are alſo equal betwixt themſelves, *Q. E. D.*

Corollary.

THIS propoſition is like to the 35th of the firſt book; for it affirms concerning ſolids, what that doth touching planes. Wherefore the demonſtration of the reſt of the caſes will be like alſo.

PROP. XXXI. Theorem.

PAralłelopipeds upon equal baſes (AO and E G) and in the ſame altitude (S) are equal. *Fig. 33.*

Firſt, let the parallelopipeds have their ſides perpendicular to the baſes. Unto the ſide F G, produced, let there be made a parallelogram G M K H, equal and like to the parallelogram A O, and the parallelogram G M P R being perfected, let the right lines P M, R G, meet K H in Q and L. And now let parallelopipeds be underſtood to be conſtituted upon G K, G Q, G P, whoſe ſides are perpendicular to the baſes, and S is their common altitude. The ſolid E G S, is to the ſolid G P S, as E G, *per* 25. *l.* 11. is to G P; that is, becauſe E G, A O are equal, by the hypotheſis, as A O to G P: that is by the conſtruction, as GK is to GP; that is, as GQ is to GP, *per* 35. *l.* 1. that is, as the ſolid G Q S is to the ſame ſolid G S P, *per* 25. *l.* 11. Becauſe therefore the ſolids E G S and G Q S have the ſame proportion to the ſolid G P S; the ſolid E G S will be equal to the ſolid G Q S; that is, to the ſolid G K S, *per* 29. *l.* 11. that is, becauſe the baſes G K, A O, are equal and like, by the conſtruction; to the ſolid A O S, as it appears from 24. *l.* 11. and even in itſelf. Which was the thing propos'd. Note, That in this reaſoning, the ſolids are ſuppos'd to be right or perpendicular ones.

Then let the given parallelopipeds E G S, A O S, have their ſides at the baſes E G, A O, oblique. Let there now be made upon E G, A O, parallelopipeds, whoſe ſides are perpendicular to the baſes in the height S; theſe will be equal to the oblique ones by the 29th and 30th. Wherefore ſeeing, by the firſt part, right parallelopipeds are

are equal betwixt themſelves, the oblique ones will be equal betwixt themſelves. *Q. E. D.*

PROP. XXXII. Theorem.

Fig. 34. *ALL parallelopipeds whatever of equal height, are betwixt themſelves as their baſes.*

Let GO and A be the baſes. Upon CO make the parallelogram OE equal to A.

Upon BC, OE, let parallelopipeds be underſtood to be erected in the altitude K; theſe therefore will be parts of one parallelopiped, BEK. Therefore the parallelopiped, OEK, is to the parallelopiped, BCK, as the baſe OE, to the baſe BC, *per* 25. *l*, 11. that is, by the conſtruction, as the baſe A is to the baſe BC. But becauſe the baſes OE and A, are equal, the parallelopipeds, OEK and AK are equal, by the foregoing. Therefore alſo the parallelopiped, AK, is to the parallelopiped, BCK, as the baſe A is to the baſe BC. *Q. E. D.*

Scholium.

THAT which hath here been ſhew'd of parallelopipeds, will be demonſtrated in the twelfth book of pyramids, *Prop.* 6. Of all priſms whatever, in *Corollary* 1. after *Propoſition* 9. Of cones and cylinders, *Propoſition* 11.

PROP. XXXIII. Theorem.

Fig. 35. *LIKE parallelopipeds (HA and CM) are in a triplicate proportion of their homologous ſides (AB, BC.)*

Let the parallelopipeds, AH, CM, be like. Therefore all their planes, by *Defin.* 9. *l.* 11. are like; and conſequently AE, by *Defin.* 1. *l.* 6. is to BC, as EB to BO; and as FB is to BG, ſo is EB to BO. Moreover the angles of the planes are alſo equal, by the ſame. Therefore, let the ſolids AH, CM, be ſo placed that the equal angles CBO, ABE, may be oppoſite, and the ſides

ſides A B, C F, may lie ſo as to make one ſtrait line; and then E B, O B will alſo lie ſo as to make one ſtrait line. Now, let ſolids be imagin'd to be conſtituted upon the planes B Q and E C, in ſuch ſort that the ſolids K B, H A, may be one parallelopepid, and K B, P O, may make one parallelopepid, and P O, C M, may make one parallelopepid likewiſe. The ſolid H A, is to the ſolid K B *per* 25. *l.* 11. as A E to B R; that is, *per* 1. *l.* 6. as A B to B C; that is, as I ſhew'd above, by the hypotheſis, as B E is to B O; that is, by the ſame, as E C is to B Q; that is, *per* 25. *l.* 11. as the ſame ſolid K B, is to the ſolid P O. Therefore the three ſolids H A, K B, P O, continue the ſame proportion. But now the ſolid K B, is to the ſolid P O, by the ſame, as the baſe B R, is to the baſe B Q. that is, *per* 1. *l.* 6. as E B is to B O; that is, as F B is to B G, as it was ſhew'd above, by the hypotheſis; that is, by the ſame, as the plane F C is to the plane B S; that is, *per* 25 *l.* 11. as the ſame ſolid P O again is to the ſolid C M. Therefore the four ſolids, H A, K B, P O, C M, are continually proportional. Therefore, by *Defin.* 10 *l.* 5. the proportion of the firſt H A, to the fourth C M, is triplicate of the proportion of the firſt H A, to the ſecond K B; that is, triplicate to the proportion, *per* 25, *l.* 11. of A E to B R; that is, triplicate, *per* 1. *l.* 6. to the proportion of the homologous ſides, A B to B C. *Q. E. D.*

[Corollary (1.) " Hence, if there be four right lines " continually proportional; as is the firſt to the fourth, " ſo is a parallelopepid deſcrib'd upon the firſt, to a paral- " lelopepid like, and in like manner deſcrib'd upon the " ſecond.

(2.) " Upon this alſo depends the moſt famous pro- " blem concerning doubling the cube; of which after- " wards, *Scholium, p.* 18. *l.* 12.

(3.) " Hence alſo is to be corrected the error of thoſe, " who ſuppoſe that the proportion of like-ſolids is the " ſame as is that of their ſides. For the cube of a line, " which is double to another line, is not only double to " the other, but as eight to one. And the cube of a " line, which is treble to another line, is not only treble " to the other cube, but contains it 27 times. For 1 : " 2 : 4 : 8 ∺ and 1 : 3 : 9 : 27 ∺, and the ſame thing is

" to be said of all like bodies whatsoever; as will appear " afterwards.

(4.) " Hence the triplicate proportion of any quantities whatsoever is the proportion of the cubes of the " same quantities. Let there be any two quantities in the " triplicate proportion of the quantities, A B, B C; they " shall also be as A B cube, is to B C cube.]

Scholium.

THAT which hath here been shew'd of parallelopepids, will be demonstrated, Book 12. Of pyramids, *Prop.* 8. Of all prisms whatsoever, *Corollary* 2. *Prop.* 9. Of cones and cylinders, *Prop.* 12. Of spheres, *Prop.* 18.

PROP. XXXIV. Theorem.

Fig. 36. *IF the parallelopepids (B M, C K) be equal, their bases and altitudes are reciprocally proportional; (that is, the base A M is to the base F K, as reciprocally the height F C is to the height A B.)*

And if their bases and altitudes be reciprocally proportional, they are equal.

Part I. First, let the sides be perpendicular to the bases. If now the altitudes of the solids, BM, CK, be equal, the thing is manifest.

If the altitudes be unequal, from the greater, F C, cut off F E, equal to B A; and through E draw the plane EL, parallel to FK. The base A M, is to the base F K, *per* 25. *l.* 11. as the solid BM, is to the solid EK; that is, because, by the hypothesis, the solids BM, CK are equal, as the solid CK, is to the solid EK; that is, by the same, as CG is to EG; that is, *per* 1. *l.* 6. as CF is to EF; that is, by the construction, as CF to BA. *Q. E. D.*

Then let the sides be oblique to the bases. Let right parallelopepids be erected upon the same bases in the same height. The oblique parallelopepid will be equal to these. Wherefore seeing these, by the first part, have their bases and altitudes reciprocal, those also will be likewise. *Q. E. D.*

Then let the ſides be oblique to the baſes. Let right parallelopepids be erected upon the ſame baſes in the ſame height. The oblique parallelopepids will, *per* 29. and 30. *l.* 11. be equal to theſe: Wherefore ſeeing theſe, by the firſt part, have their baſes and altitudes reciprocal, theſe alſo ſhall be ſo likewiſe. *Q. E. D.*

Part II. Let the altitudes be unequal, and the ſides perpendicular to the baſes; and from the greater, CF take EF, equal to AB. The ſolid BM is to the ſolid EK, *per* 32, *l.* 11. as AM is to FK; that is, by the hypotheſis, as CF is to AB; that is, by the conſtruction, as CF is to EF; that is, as CG is to (*a*) EG; that is, (*b*) as the ſolid CK is to the ſame ſolid EK. Therefore the ſolids BM and CK have the ſame proportion to EK: Therefore they are equal. *Q. E. D.*

(a) *Per* 1. *l.* 6.
(b) *Per* 25. *l.* 6.

Corollaries.

WHAT affections have been demonſtrated of parallelopepids, *Prop.* 29, 30, 31, 32, 33, 34, do alſo agree to triangular priſms, which are the halves of parallelopepids. As is manifeſt from *Prop.* 28. Therefore,

1. The triangular priſms, which are of equal height, are as their baſes AB. *Fig.* 37.

2. If they be like, their proportion is triplicate to the proportion of the ſides, oppoſite to the angles.

3. If they be equal, they reciprocate their baſes and altitudes; and if they reciprocate their baſes and altitudes, they are equal.

Scholium.

WHAT hath here, in *Prop.* 34 been ſhew'd of parallelopepids, will be demonſtrated in the 12th Book of pyramids, *Prop.* 9. Of all priſms whatſoever, *Corollary* 3. after *Prop.* 9. Of cones and cylinders, *Prop.* 15.

PROP. XXXV.

IS very long, and ſubſervient to the following propoſition, which we will demonſtrate without it.

PROP. XXXVI. Theorem.

Fig. 38. *A parallelopepid (DH) made of three proportional right lines (A, B, C,) is equal to a parallelopepid (I N,) which is made of the mean (B,) and is equiangular to the former.*

Let the base F D, of the parallelopepid D H, have the side E F equal to A, and the other side E D equal to C: And the side E G, which stands upon the base equal to B. Thus the parallelopepid D H will be made of the three right lines, A, B, C. Then let the three sides, L X, I X, X M, and consequently all the rest, of the parallelopepid I N be equal to the middle line B: And the solid angle X, equal to the solid angle E; the parallelopepid I N will be made of the mean B, and be equiangular to the former. I say also that it is equal.

For, seeing, by the hypothesis and the construction, as F E is to L X, so reciprocally, I X is to D E, the bases also (a) D F, I L, will be equal. Now, because the solid angles at E and X are equal; if they be put within one another, (b) they will coincide; and because of the equality of the right lines, E G, X M, the points M and G, will coincide. Wherefore both the solids will have one perpendicular altitude; to wit, the right line, which is let fall from the points M, G, now become one, unto the plane of the base. The solids therefore D H, I N, * are equal. *Q. E. D.*

(a) *Per* 14. *l.* 6.
(b) *Per Defin.* 11. *l.* 11.
* *Per* 31. *l.* 11.

Scholium.

WE will further observe what is of great use, that of three lines drawn into or multiplied one by another, after what manner soever, a solid of the same magnitude is produced.

ABC. CAB. BCA.
1. 2. 3.

In the present scheme, the two first letters design the base; the third, the altitude. Let us compare the first with the second.

The base A B is to the base C A, *per* 1. *l.* 6. as the side B is to the side C; that is, reciprocally, as the height B is to the height C. Therefore, by *Prop.* 34.

ABC,

ABC is equal to CAB.

In the ſame manner it may be ſhew'd that the firſt is equal to the third, and the third to the ſecond.

PROP. XXXVII. Theorem.

Parallelopepids which are like, and deſcribed in the like manner by proportional right lines, will themſelves alſo be proportional; and converſly.

This is manifeſt of itſelf. For the proportions of parallelopepids, by the 33d of this book, will be triplicate to thoſe proportions which, by the hypotheſis, are equal, which the lines have betwixt themſelves.

The converſe is manifeſt of itſelf alſo.

The propoſition is true of all ſorts of like bodies, which will appear from book the 12th, to have betwixt themſelves a proportion triplicate to that which the ſides have.

PROP. XXXVIII. XXXIX.

THESE contain nothing remarkable, and are ſcarce of any uſe.

PROP. XL. Theorem.

THIS is of a ſmall uſe, and indeed no other than the 28th propoſition in another view.

Scholium.

FROM what hath hitherto been demonſtrated, we have the dimenſion of triangular priſms, and of quadrangular or Parallelopepids; to wit, if the altitude be multiplied into the baſe. As if the altitude be of 10 feet, and the baſe of 100 ſquare feet, (now the baſe is meaſured by *Scholium, p.* 36, or 41. *l.* 1.) multiply 10 by 100, there will ariſe 1000 cubic Feet for the ſolidity of the given priſm.

The demonſtration is eaſy. For, like as a rectangle ariſeth from the multiplication of one ſide by another, ſo a right parallelopepid is produced from the height drawn into the baſe. Therefore every parallelopepid is alſo produced from the altitude multiply'd into the baſe; ſeeing by 31. *l.* 11. it is equal to a right parallelopepid, conſtituted upon the ſame baſe with the ſame height.

Then ſeeing the whole parallelopepid is produced from the height into the whole baſe; the half of the parallelopepid (that is, a triangular priſm by 28. *l.* 11.) will be produced from the altitude multiplied by half the baſe; to wit, the triangle ILK. *Fig.* 29.

THE

THE ELEMENTS OF EUCLID.

BOOK XII.
With Us the Eighth.

WHAT in the foregoing books we have endeavoured to perform; namely, to bring the elements of the mathematicks into a more eaſy and brief method, will be to be endeavour'd in this twelfth book eſpecially; the doctrine whereof is moſt neceſſary, but the demonſtrations are ſo prolix, that they commonly make beginners almoſt to deſpair. We have ſo propos'd to ourſelves to remedy this evil, that in the mean while we will not depart from the rigour of geometrical demonſtration, Which thing, whether or no we have attain'd, the reader will underſtand, if he ſhall compare this of ours with *Euclid*'s prolixity.

"Now, after *Euclid* had in the former book declared "the elements of ſolids, and defined the meaſures of the "moſt eaſy bodies, thoſe, namely, which are terminated "with plain ſurfaces; In this twelfth book he conſiders "bodies bounded with curve ſurfaces; to wit; cylinders, "cones and ſpheres; compares them betwixt themſelves; "and defines their meaſures. This book is indeed moſt "profitable, becauſe it contains thoſe principles on which "the chief maſters of geometry, and eſpecially *Archimedes*, "have built ſo many famous demonſtrations, concerning "the cylinder, cone and ſphere.

De-

DEFINITIONS.

Fig. 1. l. 12. 1. A Pyramid is a ſolid (ZL) comprehended under the triangles (ALC,) (CLF, FLB, BLA) placed from one plane (Z) to one point (L.)

The plane Z is called the baſe, and may be either a triangle or quadrangle, or any other figure, from each of the ſides whereof there ariſe triangles meeting together in the point L, which is called the vertex, or top.

As the triangle amongſt rectilinear plane figures, ſo the pyramid amongſt ſolid ones is the firſt and moſt ſimple.

Fig. 2. 3, 2. If without the plane of ſome circle (CL) there ſhall be taken the point (A,) and from it be drawn the infinite right line (AF,) touching the circle in C; and this line [the point (A) remaining fix'd] be turn'd about the circumference of the circle, until it returns thither, from whence it began to be moved; the ſurface deſcribed by the right line (ACF) is term'd a conical ſurface, and the body, which is contain'd under this ſurface, and the circle (CL) is call'd a cone.

The vertex of the cone is (A.)

The circle (CL) is the baſe of the cone.

The right line (AB,) drawn from the vertex to the center of the baſe, is the axis of the cone.

The ſide of the cone is the right line (AC) drawn from the vertex to the circumference of the baſe, which that it is wholly in the ſurface of the cone, is manifeſt from the production of the figure.

* *Fig* 2. A right * cone is, when the axis (AB) is perpendicular to the baſe.

† *Fig.* 3. A ſcalene † or oblique cone, is, when the axis (AB) is not perpendicular to the baſe.

A right cone is alſo made by a right-angled triangle (CBA) turn'd round about one of the perpendicular ſides (AB.) See *Fig.* 2.

Fig. 4, 5. 3. If an infinite right line (COF) be turn'd about two circles (CL, OQ) equal and parallel, until it returns to that place from whence it began to be mov'd, and remains always, whilſt it is mov'd, parallel to itſelf, the ſurface deſcribed by the right line (COF) is called a cylindrical ſurface; and the body which is contain'd under this ſurface, and the two circles, is call'd a cylinder.

The

The bafes of the cylinder are the circles (C L, O Q) the right line (A B) which connects the centers of the bafes, is called the axis. The right line (O C) in the furface of the cylinder, touching both the bafes, is called a fide of the cylinder.

A right cylinder, is when the axis is perpendicular to the bafe. *Fig.* 4.

A fcalene or oblique cylinder, is, when the axis is not perpendicular to the bafe. *Fig.* 5.

A right cylinder, is alfo made by a rectangle (O C B A) turned round about one fide (B A.) See *Fig.* 4.

4. Like cones and cylinders are thofe, which have their axis (A K, Z O) and the diameters of their bafes (B F, Q R) proportional. *Fig.* 20,21

5. A fphere, is a folid contained under one furface, unto which furface all the right lines that are drawn from a certain point within the figure, are equal amongft themfelves. That point is called the center. The diameter of the fphere is a right line drawn through the center unto the furface on both fides.

A fphere is produced if a femi-circle be turned about its diameter (A F) which remains in the mean while unmoved. *Fig.* 6.

6. Magnitudes infcribed into, or defcribed about, fome figure, whether they be greater or leffer than the figure, are then faid to *end* in the figure, when they will at the laft differ from it by a quantity lefs than any given one whatfoever, or how fmall foever.

Therefore if thofe magnitudes which are defcribed into fome figure, will at laft fall fhort of it by a deficiency lefs than any given one whatfoever, the magnitudes infcribed are faid to *end* in the figure; and if thofe which are circumfcribed about fome figure, will at laft exceed it by an excefs lefs than any given one whatfoever, they fhall be faid to end in the figure.

PROP.

PROPOSITION I. Theorem.

Fig. 6, 7. *THE proportion of like polygons inscribed in a circle, is duplicate to the proportion of the diameters (A F, I C.)*

Let A O, B F, I R, L C, be drawn. Because the polygons are supposed to be like, the angles (O B A, R L I) will (*per Defin.* 1. *l.* 6.) be equal; and the sides O B, B A, proportional to the sides R L, L I. Therefore in the triangles A O B, R I L (*per* 6. *l.* 6.) the angles O and R are equal. Therefore also the angles B F A and L C I, which stand upon the same arches, B A, L I, are (*per* 21. *l.* 3.) equal. But the angles, F B A, C L I, in semi-circles, are (*per* 31. *l.* 3.) right ones. Therefore the other angles (*per Corol.* 9. *p.* 32. *l.* 1.) B A F, L I C, are equal. Therefore because the triangles F A B, C I L, are equiangular to each other, they are (*p.* 4. *l.* 6.) like: and B A will be to L I, as A F to I C. Now, because, by the hypothesis, the polygons are like, their proportion will be duplicate (*p.* 20. *l.* 6.) to the proportion of the sides B A, L I; that is, as I have already shewed, duplicate to the proportion of the diameters A F, I C. *Q. E. D.*

Corollary.

Fig. 6. 7. THE circumferences of like polygons inscribed in a circle are betwixt themselves as the diameters.

Seeing it hath already been shewed, that A B is to L I, as A F is to I C; O B will also be to R L, as A F to I C: And so of the rest of the sides. Therefore all the sides together, will be to all the sides together, that is, one circumference to another, as A F is to I C.

A Lemma.

Fig. 8. POLYGONS inscribed in a circle, end in a circle. Inscribe a square, as A C B D. Seeing this is half (*per Schol. p.* 6, and 7. *l.* 4.) of the square which is circumscribed, it will be greater than half of the circle. Wherefore if this

this be taken out of the circle, there will be taken out of it more than half. Then each arch being bisected in E, K, I, H, inscribe an octagon: and let F G touch the circle in E; which F G, let B C, D A meet in G and F; C F will be a parallelogram, of which, seeing the triangle C E A (*per* 41. *l.* 1.) is half, this will be more than half of the segment C E A. In the same manner each of the triangles, A K D, D I B, *&c.* is more than half each of the segments. Therefore all the triangles are more than half all the segments. Therefore if you take these out of those, that is, out of the remainder of the circle, more than half will be taken away. In the same way of arguing, if there be inscribed in the circle, polygons of sides always double in number; I can shew that there will always be taken out of the remainder of the circle more than half. Therefore the remainder must at last be less than any given one whatsoever; and consequently the inscribed polygons will at last fall short of a circle by a quantity less than any given one whatsoever, that is, (*per Defin.* 6. *l.* 12.) will end in a circle.

PROP. II. Theorem.

THE *proportion of circles is duplicate to the proportion of their diameters.* Fig. 6, 7.

The proportion of polygons inscribed in a circle without end, is (*per* 1. *l.* 12.) duplicate to the proportion of the diameters. But polygons (by the foregoing *Lemma*) inscribed in a circle infinitely, at last end in the circle. Therefore the proportion of circles is also duplicate to the proportion of the diameters.

PROP. III. IV.

ARE *prolix, and hard for young beginners, and have no other use, than that they serve to the demonstration of the fifth, which we shall demonstrate much more easily without them.*

Lemmata,

Lemmata, or perparatory propositions to Prop. V.

Lemma I.

Fig. 9. IF two triangular pyramids be cut with planes (O S E, R X Z) parallel to the bases (A B C, I Q V) which same planes divide the sides (C F, Q L) proportionally in (E and Z, then O S E, R X Z will be betwixt themselves, as the bases (A C B, I Q V.)

Because the parallel planes, O S E, A B C, are cut by the planes B F C, A F B, A F C, the common sections, S E, B C, and O S, A B, and O E, A C (will be *per* 16 *l.* 11.) parallel. Wherefore the angles O S E, A B C, and S O E, B A C, and O E S, A C B, two and two, are (*per* 10. *l.* 11.) equal. Wherefore the sections, O S E, A B C, are like (*per* 4. *l.* 6.) In the same manner I might shew that the sections R X Z, I V Q, are like. Therefore (*per* 19. *l.* 6.) the proportion of the section A B C, to the section O S E, is duplicate to the proportion of the side B C, to the side S E; and the proportion of the section I V Q to R X Z, is duplicate to the proportion of V Q to X Z. But the proportions of B C to S E, and of V Q to X Z, are the same, for B C is to S E (by *Corollary* 1. *per* 4. *l.* 6.) as C F to E F; that is, by the hypothesis, as Q L to Z L; that is, (by the same *Coroll.*) as V Q to X Z. Therefore the proportion of A B C to O S E, is the same with the proportion I V Q to R X Z. *Q. E. D.*

Lemma I.

Fig. 10. PRISMS inscribed infinitely in a pyramid (Z C A F) which hath a triangular base, end in the same pyramid.

Let the side of the pyramid be divided into a certain number of equal parts, A B, B G, G F, and through B and G, there being made the sections, G D N and B E P, parallel to the base Z A C; let the triangular prisms B E P M A O and G D N K B Q be understood to be inscribed in the pyramid. These then being continued without the pyramid, let there be understood to be described about the pyramid the prisms C I B A, P X G B, N H F G. The excesses of the circumscribed prisms above the inscribed ones, are the solids I M, X K, H G, which taken together,

they are equal to the prism C I B A: For H G (*per* 25. 11.) is equal to D B, and consequently H G with X K, are equal to P X G B, that is, (by the same) to M E B A. Therefore the three, H G, X K, I M, are equal to the whole, C I B A. But if A F be divided without end into more equal parts, and consequently the number of prisms be infinitely encreased, AB will become less than any given line. Therefore (as it is manifest from *p.* 25. *l.* 11.) the prism C I B A will become less than any given one. Therefore the excess of their circumscribed prisms, (and much more of the pyramid Z C A F, which is part of the prisms circumscribed about it) above the inscribed prisms, will be less than any given prisms. Therefore the inscribed prisms (by *Defin.* 6 *l.* 12.) end at last in a pyramid. *Q. E. D.*

PROP. V. Theorem.

TRiangular pyramids of the same height have that proportion betwixt themselves, which their bases (A Q R, E S X) have.

Let the equal altitudes of the pyramids be represented by the side A P, E Z; which, on both sides, let be divided into as many equal parts as you will, but so that they be of the same number; and let there be made through the points of the divisions, sections parallel to the bases: Let triangular prisms, of the same number, and the same height, be understood to be inscribed in both pyramids. And now because the prisms, L A, I E, are of the same height, the prism L A will be to the prism I E (by *Coroll.* 1. *p.* 34. *l.* 11.) as the base L O B is to the base I N K; that is, by (*Lemma* 1.) as the base Q R A is to the base S X E. In the same manner I might shew that each inprisms inscribed into the pyramid Q P A R is to each inscribed into the pyramid S Z E X, as the base Q A R is to the base S E X. Therefore all of them together, are to all of them together, as base is to base. Wherefore seeing they at last end (*per Lem.* 2.) in the pyramids themselves, the pyramids themselves also will be as their bases. *Q. E. D.* *Fig.* 11.

PROP. VI. Theorem.

Fig. 12, 13. *ALL pyramids whatsoever, which are of equal height, have that proportion betwixt themselves, which their bases (AB, CFO) have.*

Let their bases be resolved into triangles, A, B, C, F, O: and the whole pyramids into triangular pyramids. The pyramid AX, is to the pyramid OZ (by the foregoing) as A is to O; and the pyramid BX, is to the pyramid OZ, as B is to O (by the same.) Therefore the pyramids AX, BX, together (that is, the whole pyramid ABX) are to the pyramid OZ, as A, B, together, are to O. By the same argumentation, the pyramid ABX, is to the pyramid FZ (by the foregoing) as AB, are to F: And ABX, is to CZ, as AB, is to C. Therefore ABX, is to the three, OZ, FZ, CZ, together; that is, to the whole pyramid, OF, CZ, as, AB, together, is to O, F, C, together. *Q. E. D.*

PROP. VII. Theorem.

Fig. 14. *EVERY pyramid is the third part of a prism, which hath the same base and height.*

First, let the triangular pyramid, BGAC, have the same base and height with the prisms, BACFEO: Let BF, AO, AF, be drawn. The triangles, BFC, BFO, are (*per* 34. *l.* 1.) equal. Therefore the pyramid, BFCA, is equal to the pyramid, BOFA. For the same reason, OEAF, is equal to the pyramid, OBAF; that is, to the pyramid, BOFA, for they are the same pyramids. Therefore BFCA, and OEAF, are also equal. Therefore all three, BFCA, OEAF, OBAF, or BOFA, are equal. Therefore the three together are triple of one BFCA. But those three constitute the prism, BACFEO. That prism therefore is triple to the pyramid, BFCA; that is, (*per* 5. *l.* 11.) to BGAC. *Q. E. D.*

Then let any pyramid whatsoever have the same base and height with the prism, AEFH: The lines BC, Fig. 15. BO, BE, and NI, NG, NH, being drawn, resolve the prisms into triangular prisms, and the pyramid into triangular

[ang]ular pyramids. Which being done, the demonstration is manifest from the first part: For each part of the prisms will be triple of each part of the pyramids. And consequently the whole prism will be triple to the whole pyramid. *Q. E. D.*

PROP. VIII. Theorem.

THE proportion of like pyramids (O A C B, K H I N) is triplicate to that which the homologous sides (A B, H N,) have to each other.

First let them be triangular: The parallelograms, A M and H Q being perfected, set upon them the parallelopepids, A G, H L, in the height of the pyramids; which seeing the pyramids are like, will also, (as appears from *Defin.* 9. *l.* 11.) be like. Then let E F, R P, be drawn; and through E F, C B, as likewise through R P, I N, the parallelopepid will be cut (*per* 28. *l.* 11.) into two equal prisms: each of which will be triple to the pyramids, O A C B and K H I N (by the foregoing.) Therefore both together, that is, the whole parallelopepids, A G, H L, will be six-fold of the pyramids. Therefore the pyramids are proportional to the parallelopepids. But (*per* 33. *l.* 11.) the proportion of these each to other is triplicate to the proportion of the sides, A B, H N. Therefore so likewise is the proportion of the pyramids. *Fig.* 16.

But if the like pyramids shall be polygonal, let them be resolved into the triangular ones, A R, B R, C R and O K, E K, F K. You may from 20. and 5. *l.* 6. and *Defin.* 9. *l.* 11. easily shew, that A R is like to O K, and B R to E K, and C R to F K. Therefore, by the former part, the proportion of the pyramids, A R, O K, is triplicate to the proportion of I M to P Z: And the proportion of the pyramids, B R and E K, is triplicate to the proportion of M X to S Z; that is, again, by the hypothesis, of I M to P Z, and the proportion of the pyramids, C R, F K, is triplicate to the proportion of X Q to S T; that is, again, of I M to P Z. Seeing therefore the proportion of each to each is triplicate to the proportion of I M to P Z, the proportion also of all to all (that is, the proportion of the whole pyramid, A B C R, to the whole, O E F K) will be triplicate to the proportion of I M to P Z. *Q. E. D.* *Fig.* 17.

PROP.

PROP. IX. Theorem.

Fig. 18, 19 *EQUAL pyramids have their bases and altitudes reciprocally proportional; and those which have them so, are equal.*

Part I, First, let the pyramids be triangular, B A C O, K H N L: The parallelograms B E, H R, being perfected, upon these set the parallelopepids, B F, H P. These will be (as was shewed in the foregoing) six fold of pyramids, which are, by the hypothesis, equal, and consequently will be equal betwixt themselves. But now the altitudes of these parallelopepids H K, B A, are the same with those of the pyramids; and the bases B E, H R, are double to the pyramidal bases, (*per* 34. *l.* 11.) B C O, H N L, and consequently proportional to them. Seeing therefore by reason of the equality of the parallelopepids, as B E is to H R, so (by the same) is reciprocally H K, to B A; it will also be that, as the base B C O is to the base H N L, so reciprocally, is the altitude H K to the altitude B A. *Q. E. D.*

But if the pyramids have polygonal bases, let them be reduced into triangular ones, retaining the same altitudes; and these will be equal to those by the sixth. But the pyramids thus reduced have, as we have now demonstrated, their bases and altitudes reciprocally proportional. Therefore the given polygonal pyramids also have their bases and altitudes reciprocally proportional. *Q. E. D.*

Part II. Because it is now supposed, that B C O is to H L N, as H K is to B A; B E will also be to H R, as H K is to B A. Therefore the parallelopepids, B F, H P, are (*per* 34. *l.* 11.) also equal. Therefore their sixth parts also, to wit, the pyramids B A C O, H K N L, are equal. *Q. E. D.*

Corollaries.

WHAT has been demonstrated of pyramids in *Proposition* 6, 8, 9. does also agree to all prisms whatsoever; seeing these are (*per* 7. *l.* 12.) triple to pyramids which have the same bases and altitudes. Therefore,

1. In prisms of the same height, their proportion is the same as that of their bases. For this was shewed of pyramids, *Prop.* 6.

2. The

2. The proportion of like priſms is triplicate to the proportion of their homologous ſides. For this was ſhew'd concerning pyramids, *Prop.* 8.

3. Equal priſms have their baſes and altitudes reciprocally proportional; and thoſe which have them ſo are equal. For this is ſhew'd of pyramids, *Prop.* 9.

It is ſtrange that theſe things were paſs'd over by *Euclid*, ſeeing they are the chief things which can be delivered concerning rectilinear ſolids.

Scholium.

FROM what has been hitherto demonſtrated is drawn the method of meaſuring any priſms or pyramids whatſoever.

The ſolidity of a priſm is produced from the altitude multiplied into the baſe; and that of a pyramid from the third part of the altitude multiplied by the baſe.

As if the altitude of a priſm be of 5 feet, but the baſe contains 25 ſquare feet; multiply 25 by 5, and there ariſe 125 cubic feet for the ſolidity of the priſm.

For let there be a polygonal priſm, as A H. And let *Fig.* 15, 16 the triangle B A C be underſtood to be equal to its baſe A E; and upon B A C, the priſm B E to be ſet at equal height with A H. The priſms B E, A H will be (by *Corollary* 1. foregoing) equal. But the priſm B E (by *Schol. p.* 40. *l.* 11.) is produced from its altitude drawn into the baſe B A C; that is, into A E, by conſtruction. Therefore the priſm A H alſo is made of its baſe A E, multiplied by its height, which is ſuppoſed to be equal to the height of the priſm B E.

From hence, and from the 7th, the demonſtration of the ſecond part is alſo manifeſt.

A Lemma to Propoſition 10.

PYRAMIDS and priſms, which are inſcrib'd in cones and cylinders infinitely, do at laſt end in the cones and cylinders.

This is demonſtrated as the *Lemma* of *Propoſition* 2. with the help of *Propoſition* 6. and of *Corollary* 1. after *Propoſition* 9, if as their planes inſcrib'd in a circle, ſo here priſms and pyramids which ſtand upon thoſe planes as their baſes, be continually taken away from the cones and cylinders.

PROP. X. Theorem.

Fig. 20. *EVERY cone is a third part of a cylinder, having the ſame baſe and height.*

Let a regular polygon of as many ſides as you pleaſe be underſtood to be inſcrib'd in the baſe C L, and upon it, as the baſe, for a cone let a pyramid, and for a cylinder, a priſm be inſcrib'd. The pyramid (*per* 7. *l.* 12.) will be a third part of the priſm. And if again in the circle a polygon of twice as many ſides be inſcrib'd, and upon it be inſcrib'd for a cone a pyramid, but for the cylinder a priſm; the pyramid will be again a third part of the priſm. And thus it will always be. Wherefore ſeeing pyramids end in a cone, and priſms in a cylinder, the cone alſo will be a third part of the cylinder. *Q. E. D.*

PROP. XI. Theorem.

Fig. 20, 21 *CONES of equal height (B A F, Q X R) are as their baſes (C L, S E.) The ſame thing belongs to cylinders of equal height alſo.*

Pyramids inſcrib'd into cones of equal height, are as their baſes, (*per* 6. *l.* 12.) But pyramids do at length end in cones. Therefore cones alſo are as their baſes. And ſeeing cylinders are three-fold of cones, which have the ſame baſe and altitude with them, they alſo will be as their baſes. *Q. E. D.*

Corollary.

IN the ſame manner it may be demonſtrated, that alſo priſms and cylinders of equal height are betwixt themſelves as their baſes; yea, that all cylindrical bodies of the ſame altitude; that is, which are produced from whatſoever planes multiplied by the ſame altitude, are betwixt themſelves as their baſes. You may reaſon in the ſame manner of pyramids and cones of equal altitudes, and of all conical bodies whatſoever.

PROP.

PROP. XII. Theorem.

THE proportion of like cones (BAF and QZR) is triplicate to the proportion of the diameters (BF and QR) which are in the Bases. The same thing is to be said of like Cylinders. Fig. 20, 21

In the bases of the like cones, let regular polygons be inscrib'd, which polygons consequently will be like. The pyramids which are inscrib'd upon these polygons will also be like; as may be easily shew'd. Therefore their proportion is triplicate (*per* 8. *l.* 12) to the proportion of the sides BL, QE; that is, to the proportion of the diameters BF, QR. Wherefore seeing the pyramids end in cones, the proportion also of the cones is triplicate to the proportion of the diameters BF, QR. *Q. E. D.*

The theorem is manifest of cylinders, seeing they are triple to cones.

PROP. XIII. Theorem.

IF a cylinder (BI) be cut with a plane (RL, parallel to the bases (BQ, CI;) one part, (BL,) shall be to the other part, (RI,) as one segment of the axis (AO) is to the other segment of the Axis (OF.) Fig. 22.

This proposition is demonstrated, as at the first of *l.* 6.

The theorem is in the same manner true of the superficies.

PROP. XIV. Theorem.

CYLINDERS (AR and CI) of equal Bases (MQ, GH) are as their Altitudes (LZ, SF.) The same thing happens to cones. Fig. 23, 24.

Cut off from the higher cylinder AR, the cylinder AO, whose height LE is the same with SF. Therefore (*per* 11. *l.* 12.) the cylinders AO, CI, are equal. Seeing therefore the cylinder AO, is to the cylinder AR, (by the

the foregoing) as L E is to L Z; C I also shall be to A R, as L E is to L Z; that is, (because L E and S F are equal, by construction) as S F to L Z. *Q. E. D.*

Corollary.

THE theorem is also true of prisms, and likewise o pyramids, and the demonstration altogether alike. Bu of prisms, the thing is demonstrated from *Corol.* 1. *p.* 9. *l* 1. and 25. *l.* 11. and its *Corol.* Of pyramids from this, an from *p.* 7. *l.* 12.

PROP. XV. Theorem.

Fig. 24, 25 *EQUAL cylinders (A R, D F) have their base and altitudes reciprocally proportional: an if they have them so, they are equal. The sam thing is true of cones.*

This is demonstrated, as *Prop.* 34 *l.* 11. only for 32, an 25. *l.* 11. there cited, there must be cited here *Propositio* 11. and 13. *l.* 12.

Scholium.

Fig. 25, 24 WHEREAS *Euclid* hath said nothing of compound proportion in bodies, we shall briefly demonstrate it in this place.

1. A cylinder hath to a cylinder, and a prism to a prism, a proportion compounded of the proportions of the bases and altitudes.

Let F D and A R be cylinders of different altitudes (for in those of equal altitude the thing is manifest.) From the higher, cut off A O of equal height with F D. And let the proportion be thus; as the base U T is to the base M Q, so F N to X; and as the altitude N D or B O is to the altitude B R, so is X is Z. We must therefore shew, that the cylinder F D is to the cylinder A R, as F N is to Z. The cylinder F D is to the cylinder A O (*per* 11. *l.* 12.) as the base V T is to the base M Q; that is, (by construction) as F N is to X; but the cylinder A O is to the cylinder A R (*per* 13. *l.* 12.) as B O to B R, that is,

(by construction) as X to Z. Therefore, by proportion of equality, the cylinder F D is to the cylinder A R, as F N to Z.

The proposition may be demonstrated in the same manner of prisms, but from *Cor.* 1. *p.* 9, and *Cor.* *p.* 14.

2. A cone hath also to a cone, and a pyramid to a pyramid, a proportion which is compounded of the proportions base to base, and altitude to altitude.

For (by *Prop.* 10. and 7. *l.* 12) they are third parts of linders and prisms.

PROP. XVI, XVII.

THESE propositions, the most prolix of all other, have no other use, than to serve to the monstrating Prop. 18, *we shall demonstrate another more easy way.*

Lemma to Proposition 18.

CYLINDERS inscrib'd in an hemisphere end in the hemisphere. Let P Z Y be the greatest semi circle of the hemisphere; and let the radius A Z be perpendicular to the diameter P Y. Cut A Z into a certain number of equal parts, A M, M N, N Z; and there being drawn through the points of the divisions M, N, the perpendicular lines B O, *&c.* Let there be inscrib'd in the semicircles the rectangles O B R K, E D H S; which afterwards being continued without the semi-circle, let there be understood to be describ'd about the semi-circle the rectangles F T Y P, L V B O, Q X D E. They will all of them be of the same height, and the excesses of the circumscribed ones, above those which are inscrib'd will be the planes F K, L S, X E, V H, T R, which, taken together, make the rectangle F T Y P. For because X E, is equal to D S, those L S, V H, X E, together, will be equal to the rectangle L B, that is, O R. Wherefore if you add on both sides the planes F K, T R, all those, FK, L S, XE, V H, T R, taken together, will be equal to the rectangle F T Y P. If now the semi-circle, with the rectangles, be understood to be turn'd about the radius A Z, which is in the mean while unmov'd, the inscribed rect- *Fig.* 26.

angles E H, O R, will produce cylinders inſcribed in the hemiſphere; and the circumſcribed rectangles will produce cylinders circumſcribed about the hemiſphere, ſtanding one upon another; and as the exceſſes of the circum ſcribed rectangles above the inſcribed ones, was the re angle F Y; ſo likewiſe the exceſſes of the circumſcri cylinders above the inſcribed ones, will be the cylin which is produced from the rectangle F Y. But now altitude of this cylinder will be made leſs than any giv height; and conſequently (as is manifeſt from 13. *l.* 1 itſelf will grow to be leſs than any given cylinder, if radius being divided into more equal parts without e the number of rectangles, and from thence of cylinde be infinitely encreaſed. Therefore the exceſs of the c cumſcribed cylinders, and much more of the hemiſph itſelf, which is only a part of the circumſcribed cylind above the inſcribed ones, will at laſt become leſs than a given one. Therefore (by *Defin.* 6. *l.* 12.) cylinders ir nitely inſcribed in an hemiſphere, do at length end in t hemiſphere itſelf. *Q. E. D.*

Corollary.

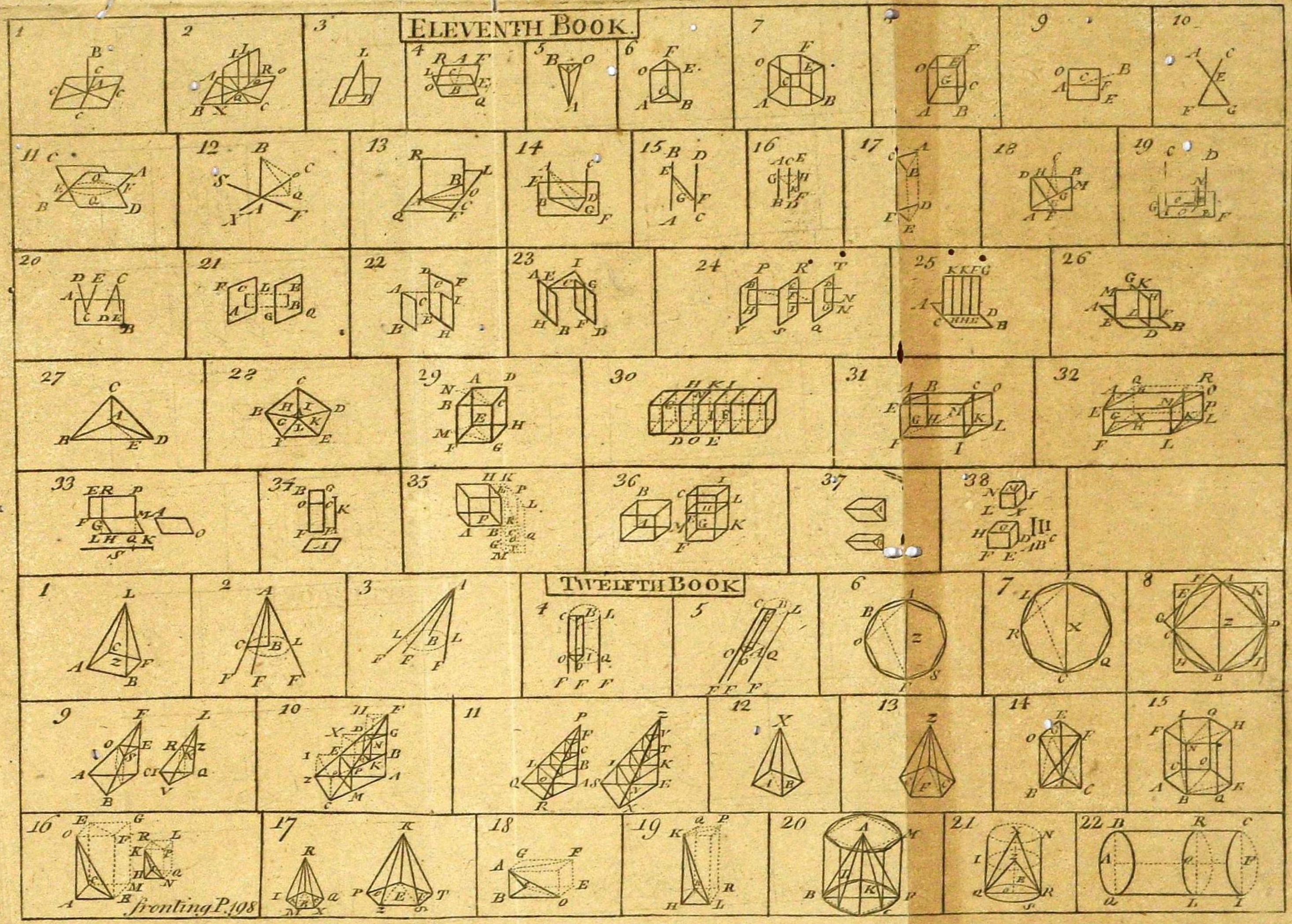
IN the ſame manner it will be demonſtrated, that cyli ders inſcribed in a cone, conoid, ſpheroid, *&c.* do at laſt end in the ſame.

PROP. XVIII. Theorem.

Fig. 27.

THE proportion of ſpheres is triplicate to the proportion of their diameters (*B K, R Z.*)

The radiuſes A B, YR, being divided into as many equal parts as you will, but of an equal number, and there being drawn through the points of the diviſions perpendicular, *&c.* Let rectangles of an equal number be underſtood to be inſcribed in the greater ſemi-circles of the ſpheres, which rectangle being turn'd about, the unmov'd radiuſes A B, Y R, will be conceiv'd to inſcribe in both the hemiſpheres, a like number of cylinders ſtanding one upon another. Now, becauſe K C is (*per Cor. p.* 3. *l.* 6) to C F, as C F is to C B; the proportion of K C to C B (by *Defin.*


ELEVENTH BOOK.
TWELFTH BOOK
fronting P.198

Defin. 10. *l.* 5.) will be duplicate to that of FC to CT, that is, to the proportion of FC to CB. In like manner the proportion of ZE to ER will be duplicate to the proportion of XE is to ER. But, by the construction, KC is to CB, as ZE is to ER. Therefore FC also is to BC, as XE to ER. But BC, by the construction, is to CO, as RE to ES. Therefore by equality, FC is to CO, as XE is to ES. Therefore (by *Defin.* 4. *l.* 12.) the cylinders FL, XQ, are like, and consequently their proportion is (*p.* 12. *l.* 12.) triplicate to the proportion of their diameters, FI, XV, or of the semi-diameters, FC, XE, which are the bases. But the proportion of FC to XE, is the same with the proportion which is betwixt the diameters of the spheres BK, RZ; (for, as I have already shew'd, FC is to XE, as CO is to ES; that is, as BK is to RZ, which, by the construction, are equi-multiples of those, CO, ES.) Therefore the proportion of the cylinders, FL, XQ, is triplicate to the proportion of the diameters, BK, RZ. In the same manner we might demonstrate that each cylinder inscribed in one hemisphere, bears to each cylinder inscribed in the other hemisphere, a proportion triplicate to the proportion of the diameters, BK, RZ. Therefore also the proportion of all together, to all together, is triplicate to the proportion of the diameters BK, RZ. Wherefore seeing the aggregates of the cylinders do at length end in their hemispheres, the proportion of the hemispheres also, and consequently of the spheres, will be triplicate to the proportion of their diameters. *Q. E. D.*

Corollary.

THEREFORE the proportion of the diameters being known, the proportion of the spheres becomes known likewise. As if the diameter of the lesser be one foot, that of the greater ten feet; let the proportion of one to ten be continued through four terms, 1, 10, 100, 1000; as 1, the first is to 1000, the fourth term, so is the lesser sphere to the greater.

The dimension of cones, cylinders and of the sphere, will be exhibited in the following book out of *Archimedes.*

Scholium,

Scholium.

AS like plain figures are encreas'd or diminish'd in any given proportion by one mean proportional, so like bodies are encreased or diminished by two mean proportionals.

Let a sphere, or cube, or any other body whatsoever, be given, whose radius, or side, is A. Likewise let any proportion whatsoever of A to B be given, as the double, or 2 to 1. A body is to be discovered both double to the given one, and like to it.

Betwixt the terms of the given proportion A and B, let there be found two mean proportionals X, Z, according to what was taught in the *Scholium* of *Prop.* 13. *l.* 6. A sphere whose radius is X, or other body like to the given one, which is made upon the side X, will be double to the given one.

For like bodies, whose radius's, or sides, are A and X, have betwixt themselves a proportion which is triplicate to the proportion of A to X, (by *Corollary*, *Proposition* 9. and by *Proposition* 12. and 18. *L.* 12.) that is, the same (*per Defin.* 10 *l.* 5.) which A hath to B.

And this is that most celebrated problem which, from *Apollo* and *Delos*, is called the *Deliacal* problem; because at the time of a most grievous pestilence, which wasted *Athens*, being consulted, he gave answer, that the pestilence would cease, if his altar, which was of a cubical form, were doubled. Thus *Valerius Maximus*, *L.* 8.

THEOREMS

SELECTED OUT OF

ARCHIMEDES:

By ANDREW TACQUET,

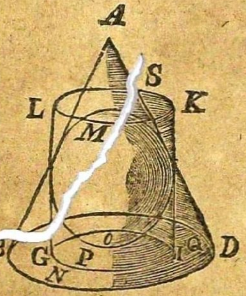
OF THE

SOCIETY of JESUS;

And Demonſtrated in a more Eaſy and Compendious Way.

To which are added,

Some other agreeable Propoſitions, newly invented, by the ſame ANDREW TACQUET.



Una tribus Ratio eſt.

DUBLIN:

Printed in the Year, M,DCC,XCI.

TO THE

READER.

ALBEIT there have appeared very many most excellent and admirable men in the mathematical sciences; yet the chief glory of all hath always, by a certain common consent, been given to *Archimedes* of *Syracuse*. Tho' indeed, more there are who commend, than who read him; more who admire, than understand him. The causes of which neglect seem to be these, the bulk and scarceness of copies, some obscurity of the translation, which is directly made out of the *Greek* language, together with the prolixity and difficulty of his demonstrations. I judged therefore that it would be for the profit of studious learners, if after my illustration of the elements, I should subjoin these *Theorems* which had been selected by me out of *Archimedes*, and demonstrated in a much easier and briefer way. Furthermore, I have selected those which bring along with them both more of admiration and of benefit, and have in my demonstration took such a method, that, I hope, he who understands the elements, will, without any great labour, comprehend these most excellent inventions of the prince of geometricians. I have also added at the end, thirteen propo-

sitions

[...]ons, and thereby enla[rge] the doctrine of *Archimedes* concerning the sphere and cylinder: where, amongst other things, I demonstrate, that the sesquialtera proportion is continued in the three bodies, a sphere, a cylinder and equilateral cone; both the latter being inscrib'd about the sphere. Moreover, I have added divers things here and there, amongst which the 12th *Proposition*, and the *Corollar[ies] of Proposition* 14. are the chief; and several scholiums. Make use of these discoveries, whosoever thou be'st, that art a candidate of *Geometry*; and how much thou hast improv'd in *Euclid*, make proof of in *Archimedes*. And when thou perceivest thy self to be fix'd and rais'd upwards in the contemplation of the most noble truths, raise up thy mind, while it is thus already lifted up from these lower things, yet higher, and direct it to that Truth, which is original, eternal, immense and is no other than *GOD*; by the ineffable vision of whom, I trust we shall hereafter be made eternally happy. Farewel.

THEO-

THEOREMS

Selected out of

ARCHIMEDES.

DEFINITIONS.

Or an Explanation of certain Terms.

LET there be a circle, BECG, whose center is A, its diameter BC, which let the right line EG cut at right angles, (but not through the center) in D. Let there be drawn from the center the radiuses AE, AG. This being suppos'd, *Fig. 23. Of the Table out of Archimedes.*

NOTE, (1.) That a sector of a sphere is that which is produced from the sector of the circle AECG, or AEBG, turn'd round about the diameter BC.

2. That a segment, or portion of a sphere, is that part of it which is produced from the segment of the circle ECG or EBG turn'd round about the same diameter BC.

3. The vertex, or top of the spherical portion EBG, is the extremity B of the unmov'd diameter; the basis, the circle describ'd by EG; the axis, that part of the diameter BD, which is intercepted betwixt the top B, and D the center of the base.

4. When I name the superficies of a spherical portion, or of a body inscrib'd in it, or of a cone, I always understand it without the base; and when I say the superficies of a cylinder, I mean likewise without the bases; unless the word (*whole*) be adjoin'd to (superficies;) for then the bases also are to be taken in.

Again, when I treat of cylinders or cones, I speak of no other than right ones.

Axioms.

Axio

Fig. 1, 16. 1. THE circuit of a polygon inſcrib'd in a circle is leſs than the circumference of the circle.

Fig. 1. 2. The circuit of a polygon deſcrib'd about a circle is greater than the circumference of the circle.

Fig. 16. 3. And if a polygon inſcrib'd in a circle, be turn'd about the diameter (A E) together with the circle, the ſuperficies of the body produced by the polygon, will be leſs than the ſuperficies of the ſphere. And if a polygon circumſcrib'd about the circle, be turn'd about the diameter, together with the circle, the ſuperficies of the body produced by the polygon, will be greater than the ſuperficies of the ſphere.

Fig. 17. 4. In like manner the circuit of a polygon inſcrib'd in a ſegment of a circle (D A F) is leſs than the circumference of the ſegment. And if a polygon inſcrib'd in the ſegment, be together with the ſegment (A O) turned round, the ſuperficies of the body produced by the polygon will be leſs than the ſuperficies of the ſpherical portion D A F.

Fig. 3, 6. 5. The ſuperficies of a priſm inſcrib'd in a cylinder, is leſs than the ſuperficies of the cylinder; but the ſuperficies of the priſm which is circumſcrib'd is greater.

Fig. 4, 8. 6. And the ſuperficies of a pyramid inſcrib'd in a cone, is leſs than the ſuperficies of the cone, but the ſuperficies of a circumſcrib'd pyramid is greater.

PROPOSITION I, II.

ARE *not neceſſary.*

PROP. III. Theorem.

THE circuits or polygons circumſcrib'd about and inſcrib'd in a Circle, do at laſt end in the circumference of the circle. In like manner the polygons themſelves do at laſt end in the circle.

Fig. 1. *Table Archimedes.* If, to wit, the arches being biſected without end, more and more ſides be circumſcribed about and inſcrib'd in the circle.

Part

Part I. Let there be understood to be inscrib'd in and describ'd about a circle, regular polygons; w[illegible] be done so as is set down, Prop. 12. l. 4, or [illegible] the prese[nt] figure, the thing will be the same. It is manifest (per Cor. 1. p. 4. l. 6.) that F I is to C E (that is, the whole circuit circumscrib'd, unto the whole circuit inscrib'd) as I A is to C A. But I C, the excess of the right lines I A above C A, becomes at length less than any given line, if more and more sides be understood to be infinitely circumscrib'd and inscrib'd; therefore also the excess of the circuit circumscrib'd above that which is inscrib'd, will at length become less than any given line. Therefore much more the excess of the circuit circumscrib'd above the circumference of the circle, will be less than any given one. In like manner, because I have already shew'd the defect of the circuit inscrib'd, whereby it falls short of that which is circumscrib'd, to be less than any given line: Therefore much more will the defect of the circuit inscribed, whereby it falls short of the circumference of that circle, become less than any given line. The circuits therefore, as well that which is inscrib'd, as that which is circumscrib'd, do at length (Def. 6. l. 12.) end in the circumference. Which was the first part. To demonstrate these things further is not worth the while, seeing they are manifest enough.

Part II. Because it hath already been shew'd that the excess F I, above the side E C, becomes at length less than any given line (F I is to E C, as I A to C A;) therefore also the excess of the square of F I, above the square of E C, will become at length less than any given one. But as the square of FI, is to the square of EC, so (per 20 l. 6.) is the polygon circumscrib'd, to that which is inscrib'd. Therefore the excess of the polygon circumscrib'd above that which is inscrib'd, will also become at length less than any given one. Therefore much more will the excess of the polygon circumscrib'd above the circle, become at last less than any given one; and consequently, the defect also of the polygon inscrib'd, whereby it falls short of the circle, will at length become less than any given defect. Therefore polygons, as well inscrib'd as circumscrib'd, do at last (Defin. 6. l. 12.) end in the circle. Which was the second part.

PROP.

PROP. IV. Theorem.

(c) Per Defin. 3. 4. Fig. 1.

A Regular (c) polygon (FINTR) circumscrib'd about a circle, is equal to a triangle whose base is the circuit of the polygon, and its height the radius of the circle.

And a regular polygon inscrib'd in a circle is equal to a triangle, which hath for its base the circuit of the polygon, and for its height the perpendicular (AO) let down into one side from the center.

Part I. The radius AB, drawn to the point of contact, is (*per* 18. *l.* 3.) perpendicular to the tangent IF. Wherefore, if the right lines AF, AI, AN, &c. being drawn, the polygon be resolv'd into triangles; the radius AB will be the common altitude of all; and consequently it is manifest that the triangles are equal. Therefore a triangle, which hath its base equal to the sum of the sides FI, IN, NT, &c. and AB for its altitude, will (as is manifest from 1. *l.* 6) be equal to them all, that is, to the whole polygon circumscrib'd.

Part II. This may be concluded by the same reasoning as the other.

PROP. V. Theorem.

Fig. 2.

A Circle is equal to a triangle, which hath for its base the circumference, and for its height the semidiameter of the circle.

Regular polygons circumscrib'd about a circle, and triangles which have for their bases the circuit of the polygon, and for their altitude the radius of the circle are always (by the foregoing *Proposition*) equal. But polygons circumscrib'd infinitely about the circle end in the circle, (by the third of this book; and in like manner triangles (as I will shew by and by) which have for their bases the circuit of the circumscrib'd polygon, and for their altitude the radius AB, at last end in a triangle, which hath the circum-

circumference for its base, and for its altitude the radius A B. Therefore (by the first) a circle and a triangle which hath the circumference for its base, and the radius for its altitude are equal.

But that triangles contain'd under the circuit of the polygon, and the radius of the circle, end at last in a triangle, which is contain'd under the circumference and the radius, I thus shew. Triangles under the circuit of the circumscrib'd polygon, and the radius A B, are to the triangle, which is under the circumference and the radius A B, (by 1. *l.* 6.) as base to base, that is, as the circuit of the polygon to the circumference; since this triangle and the other have a common altitude. But the circuit of the polygon (by the third) ends in the circumference. Therefore the other triangles end in this.

Corollaries.

1. FROM this and 41. *l.* 1. it is manifest that a rectangle under the radius, and half the circumference, is equal to the circle; that one under the radius, and the whole circumference, is double; that one under the whole circumference and the whole diameter is quadruple thereto.

2. A circle is to an inscrib'd square, as half the circumference (CDE) is to the diameter; but to a square circumscrib'd, as the fourth part of the circumference is to the diameter. *Fig.* 5. *l.* 4.

For the rectangle under C D E, and the radius C A or C F, that is, (by the foregoing *Corollary*) the whole circle is to the rectangle G F C E, to wit, the rectangle under F C and C F (that is, to the inscrib'd square B C D E) as (*per* 1 *l.* 6.) C D E, half the circumference is to F G or C E the diameter; which was the first thing. And consequently the circle is to the double the rectangle G F C E, (that is, to F H, the circumscrib'd square) as C D E is to the double of the diameter C E, or as the quadrant C D to the diameter C E.

3. " Of figures which are of equal circumferences, the circle is the most capacious. Let the circumference, " of any polygon whatsoever (as for instance of a square) " E G H I be equal to the circumference of the circle. I " say, that the area of the circle is greater than that of " the polygon. For the area of the circle is equal to a *Fig.* 30.

O " triangle,

“ ...gle, whoſe baſe is the circumference, and it's al-“ titude the ſemi-diameter F A: And the area of the po-“ lygon is equal to a triangle, whoſe baſe is the compaſs “ of the polygon; which, by the hypotheſis, is equal to “ the circumference of the circle, and which hath for its “ altitude the perpendicular F O, let down from the cen-“ ter of the circle unto the ſide of the polygon, which, “ ſince it is always leſs than the radius of the circle, it is “ manifeſt that the area of the polygon is leſs than the “ area of the circle. *Q. E. D.* And in like manner, “ amongſt all ſolid figures contain'd under equal ſurfaces, “ the ſphere may be demonſtrated to be the moſt capa-“ cious.]

PROP. VI. Theorem.

THE *circumference of a circle contains the diameter leſs than thrice and one ſeventh (or $\frac{10}{70}$;) and more than thrice and $\frac{10}{71}$.*

For the demonſtration of this theorem, *Archimedes* aſſumes regular polygons, one circumſcribed about a circle, the other inſcribed, and both of them of 96 ſides. And then he ſhews that the 96 inſcribed about a circle, do contain the diameter leſs than thrice and one ſeventh, and conſequently that the circumference which is leſs than them, doth alſo contain the diameter leſs than thrice and one ſeventh. But the 96 ſides inſcribed in the circumference, (and conſequently the circumference alſo which is greater than them) doth contain the diameter more than three times $\frac{1}{7}$. But this demonſtration is too long to be brought in this place. Nay, if we minded to extend our geometrical reaſoning to polygons of more ſides ſtill, we may contract the limits even now ſet, more and more without limit, and ſo come nearer and nearer for ever to the true proportion. This hath been perform'd by *Ludolph à Ceulen*, *Grimberger*, *Metius*, *Snellius* and others. The chief proportions hitherto found I ſhall here ſubjoin.

“ [However, ſince a tangent of 30 degrees, multiplied “ by 12, gives the circuit of a circumſcribed hexagon; “ and a ſine of 30 degrees multiplied by 12, gives the “ circuit of an hexagon, which is inſcribed: Foraſmuch “ alſo as in like manner the tangent of half a degree mul-

“ tiplied

"tiplied by 720, yields the circuit of a circumscribed "polygon of 360 sides; and the sine of half a degree, "the circuit of an inscribed polygon of 360 sides; and "so on for ever: It will not be difficult to understand, by "what means many such numbers may be found, out of "the now given tables of sines and tangents.]

The first proportion, which is that of *Archimedes*, is thus:

The diameter 7.
The circumf. is 22: which is greater than the true.
The diameter 71.
The circumf. is 223; less than the true one.

The proportion of 22 to 7, and 223 to 71, if they be reduced to a common consequent, (which is done after the same manner, in which fractions are reduced to the same denomination) will be thus, 1562 to 497, and 1561 to 497.

Therefore the diameter being suppos'd 497 parts, the circumference greater than the true one, will be 1562; and the circumference less than the true, 1561.

Both of them therefore differ from the true, by a quantity less than $\frac{1}{497}$ part of the diameter. But if the proportion of 7 to 22, and 71 to 223, be reduced to a common consequent, there will arise the proportions of 1561 to 4906, and of 1562 to 4906.

Therefore the circumference being suppos'd to be 4906 parts, the diameter less than the true, will be 1561, the diameter greater than the true, 1562.

Both therefore differ from the true diameter by a quantity less than $\frac{1}{4906}$ part of the circumference.

The proportion delivered by *Metius* is much more accurate than this of *Archimedes*. According to this,

The diameter is 113.
The circumference 355.

Amongst all proportions consisting of small numbers, none comes nearer to the true one; for from this, the diameter being suppos'd of 10,000,000 parts, the circumference comes to be of 31,415,929, which differs from the true one only in the first figure 9, and this by an excess, but a little greater than two ten-millioneth parts of the diameter.

 But

But more exact than both, is that double proportion Ludolphus à Ceulen; the former of which consists of 21 figures, and the latter of 36.

The diameter.
100,000,000,000,000,000,000.
The circumf. greater than the true.
314,159265,358979,323847.
The circumf. less than the true.
314,159265,358979,323846.

The difference of both the circumferences is one particle of the diameter denominated from a number which consists of unity and 20 ciphers; and consequently, as well this as that, differs from the true circumference by a quantity less than is the said small part of the diameter; to wit, one hundredth of a millioneth of a millioneth of a millioneth part.

The diameter.
100000,000000,000000,000000,000000,000000.
The circumf. greater than the true.
314159,265358,979323,846264,338327,950289.
The circumf. less than the true,
314159,265358,979323,846264,338327,950288.

The difference of both the circumferences, betwixt which is the true one, is that small part of the diameter, denominated from a number which consists of unity and 35 ciphers; which small part bears a less proportion to the whole diameter, than one little grain of sand doth to the whole globe of the earth. For the whole globe of the earth doth not consist of so many little grains of sand, as are the little parts of the said sort which are contain'd in the diameter.

It is needless to go any further. Nevertheless you may proceed infinitely, if you be minded to continue geometrical reasoning, an expedite method of which is delivered by *Snellius*.

[*The circumference being suppos'd of*
1,000000,000000,000000,000000,000000,000000 *Parts.*
The diameter will be as near as may be, of
0,318309,886183,790671,537767,926745,028724 *Parts*]

Scholium.

Scholium.

THE most excellent advantages of the proportion now delivered, are these which follow.

The invention of the diameter from the circumference.

SET the greater term of one of the proportions which have been now delivered in the first place, the lesser in the second, the circumference in the third; by these three numbers let there be sought by the golden rule a fourth proportional. That is the diameter sought.

As if the circumference of the greatest circle of the earth be suppos'd to contain 25000 *English* miles of 5280 feet each, and the diameter be sought; the terms will stand thus,

355—113—25000—7854.

Multiply now the second by the third, and divide the product by the first; and there will arise 7854 miles for the diameter of the globe of the earth.

The finding out of the circumference from the diameter.

LET the lesser term of one of the proportions above delivered be set in the first place; the greater in the second; the known diameter in the third: And by these three numbers, let there be sought a fourth proportional. That will give the sought circumference.

As if the diameter of the globe of the earth be suppos'd to contain 7854 *English* miles; and the circuit is sought; the terms will stand thus,

113——355——7854——25000.

Then multiply the second by the third, and divide the product by the first; there will arise 25000 miles for the circumference of the globe of the earth.

How little this circnmference exceeds the true one, was said above; to wit, by an excess but a little greater than are two ten-millioneth particles of the earth's diameter; that is, by 9 or 10 feet. But if we use the *Ludolphin* proportion, even the former, the terms whereof consist

of 21 figures; there will be found a circumference insensibly differing from the true, not only when the given diameter is of 7854 miles, such as is the diameter of the earth; but also although the diameter be suppos'd, of an 100 millions of those miles. For this being suppos'd, there will arise a circumference differing from the true one by a quantity about one hundred millioneth part of a foot. But if to find out the circumference of the globe of the earth, we make use of the proportion of *Archimedes*, the interval of the two circumferences, the one greater, the other less than the true one, will exceed 20 miles. *Archimedes*'s proportion therefore is not to be used but in small measures; nay, it will always be expedient to use that of *Metius*, which both consists of small terms, and is above 1000 times more exact.

The measuring of a circle.

THE semi-diameter multiplied by half the circumference, produceth the area of the circle; as is manifest from *Corollary* 1. *Proposition* 5. of this book.

As if the semi diameter of the earth, which contains 3927 miles, be multiplied by half its circumference, to wit, by 125000, there will arise 49075500 miles square for the area of the greatest circle of the earth. The difference of the circular area thus found from the true is had, if the difference of half this found circumference from the true half circumference be multiplied by the given semi-diameter; or the difference of this semi-diameter from the true, be multiplied by the given semi-circumference.

The mensuration of cylinders and cones.

I Put this here, because it depends upon the mensuration of a circle. A cylinder therefore, and any prism whatsoever, is produced from the altitude multiplied by the base: A cone and pyramid, from the third part of the altitude, multiplied by the base; for they are third parts of cylinders and prisms, having the same base and altitude with them by 10, and 7 *l.* 12.

Let the base of a cylinder or cone, be of 50 square feet, and the height of 100 feet. Multiply 100 by 50, and there arise 5000 cubic feet for the solidity of the cylinder.

der. Multiply the third part of the altitude 100, which is $33\frac{1}{3}$ by 50, there arise $166\frac{2}{3}$ cubical feet for the solidity of the cone.

PROP. VII. Theorem.

THE circumferences of circles have the same proportion betwixt themselves which their diameters have. Fig. 6, 7. l. 12.

For the circuits of like polygons, which may be inscribed in a circle without end, are always betwixt themselves as the diameters A F and I C (by *Corollary*, *p.* 1. *l.* 12.) But these circuits (by the 3d *Proposition* of this Book) end at length in the circumference. Therefore their circumferences also are betwixt themselves as their diameters. *Q. E. D.*

PROP. VIII. Theorem.

THE superficies of a prism, as well that which is circumscrib'd about, as that which is inscrib'd in a cylinder, is equal to a rectangle, whose height is the side of the cylinder, but its base equal to the circuit of the base of the prism.

Part I. The superficies of the circumscribed prism touches the cylinder according to the lines E A, N F, &c. which are the sides of the cylinder; but these (because by the hypothesis the cylinder is a right one) are right to the plane of the base, and consequently right also (by *Definition* 3. *L.* 11.) to the lines C G, G M, &c. But they are also equal betwixt themselves. Therefore one side of the cylinder is the common height of all the rectangles C O, O M, M H, &c. Therefore the superficies of the circumscribed prism is equal (as is manifest from 1. *L.* 6.) to a rectangle contain'd under the circuit of the base of the prism, and the side of the prism or cylinder. Fig. 3.

Part II. The Reason of this is the same. For the side of the cylinder is again the common altitude of the rectangles B D I K, K I P Q, &c. which constitute the superficies of the inscribed prism.

PROP.

PROP. IX. Theorem.

Fig. 4. *THE superficies of a regular pyramid circumscrib'd about a right cone, is equal to a triangle, which hath for its base the circumference* (*FHLD*) *of the pyramidal base, but its height the side of the cone* (*BG.*)

And the superficies of a regular pyramid inscribed in a right cone, is equal to a triangle, which hath for its base the circumference of the pyramidal base, but for its height the perpendicular (*BO*) *let down from the top unto a side of the base.*

Part I. Let there be drawn unto the contacts G, K, M; the right lines BG, BK, BM. These will all be sides of a right cone, and consequently equal. And because, (by the hypothesis) the axis BA is perpendicular to the plane of the base FKD, the plane also GBA (*per* 18. *l.* 11.) will be perpendicular to the plane FKD. But HG (*per* 18. *l.* 3.) is perpendicular to AG, the common section of the planes FKD and GBA. Therefore HG (as is gathered from *Defin.* 4. *l.* 11.) is also perpendicular to the plane GBA. And consequently is also perpendicular to BG. Therefore the side GB of the cone, is the height of the triangle FBH. In the same manner the side of the cone will be the height of the rest, HBL, LBD, &c. Therefore the triangle comprehended under the circumference FHLD, and the side of the cone is equal to the superficies of a pyramid circumscribed, without the base. Which was the first part.

II. The demonstration of this part is almost the same with that of the former.

PROP. X. Theorem.

THE superficies of a regular prism circumscribed about a right cylinder, ends (Def. 6. l. 12.) *in the superficies of the cylinder; and the superficies*

ficies of a pyramid circumſcrib'd about a right cone ends in the ſuperficies of the cone.

Part I. The ſuperficies of regular priſms deſcribed about and inſcribed in a cylinder without end will have at laſt a difference betwixt themſelves leſs than any which can be given. Much more therefore will the ſuperficies of a circumſcribed priſm differ from the ſuperficies of the cylinder, which is in the middle between the inſcribed and circumſcribed ſuperficies, by a difference leſs than any given one whatſoever; that is, (*Def.* 6. *l.* 12.) will end in the cylindrical ſuperficies, whilſt it continually exceeds it leſs and leſs. *Fig.* 3.

Part II. This may be ſhewed in the ſame manner from the 9th and 3d of this. *Fig.* 4.

In the figures there are only exhibited the halves of the cylinder and cone, leſt a multitude of lines ſhould breed confuſion. But the cylinder and cone are to be conceiv'd in the mind entire, and as having theſe circumſcribed priſms and pyramids encompaſſing them. For thus it more clearly appears that plain ſurfaces circumſcribed are greater, according the third *Axiom*.

A lemma to the following propoſition.

LET AB, CD, EF, be proportional, and let KB be half AB, and EG, double EF; KB, CD, EG, will alſo be proportional. *Fig.* 7.

The right line KB is to AB, as EF is to EG. Therefore the rectangle KB, EG (*per* 16. *l.* 6.) is equal to the rectangle AB, EF. But this (by 17. *l.* 6.) is equal to the ſquare of CD. Therefore alſo the rectangle KB, EG is equal to the ſquare of CD. Therefore (by 17. *l.* 6.) KB, CD, EG are proportional.

PROP. XI. Theorem.

A Circle, whoſe radius (GH) is a mean proportional betwixt the ſide of a right cylinder (BC) and the diameter of the baſe (BD) is equal to the cylindrical ſuperficies. *Fig.* 5, 6.

Let the regular, and conſequently like polygons, NM, RS, be underſtood to be circumſcribed about the circles ABN, GPH; and upon the polygon NM, let a priſm be

be conceiv'd to be erected, with which the cylinder is circumscribed. Because B D, G H, B C are, by the hypothesis, proportional, A D also, (or A N) G H, and the double of B C will, by the *Lemma*, be proportional. Now the triangle contain'd under A N, and the circuit of the polygon M N is equal to the polygon circumscribed N M, (by the fourth of this book.) And the rectangle under B C, or E F, and the same circuit N M, (that is, as is manifest from 41. *L.* 1. the triangle under the circuit N M and the double of B C) is equal (by the eighth of this book) to the superficies of a prism circumscrib'd about the cylinder. But a triangle under the circuit N M and A N, is to the triangle under the circuit N M, and the double of B C (by 1. *l.* 6.) as A N is to the double of B C. Therefore the polygon N M also is to the superficies of a prism circumscrib'd about a cylinder, as A N is to the double of B C. But because I have already shewed A N, G H, and the double of B C to be proportional, the proportion of A N to the double of B C is (by *Def.* 10. *l.* 5.) duplicate to the proportion of A N to G H. Therefore the polygon N M hath to the superficies of the prism a proportion duplicate to the proportion of A N to G H. But the polygon N M hath also to the polygon like to it, G R Q S, a proportion duplicate to the proportion of A N to G H, as is easily gathered out of 1. *l.* 12. Therefore the polygon N M hath the same proportion to the superficies of the prism, which it hath to the polygon G R Q S; which consequently is equal to the superficies of the prism. In the same manner I might shew, that the prismatic superficies, which are circumscriptible infinitely about the cylinder, are always equal to the polygons which may be circumscribed infinitely about the circle G P H. Wherefore seeing both the prismatic superficies (by the 10th of this) end in the surface of the cylinder, and the polygons in the circle G P H (by the 3d of this) the superficies of the cylinder also will be equal to the circle G P H. *Q. E. D.*

From this admirable theorem, a circle is presented, which is equal to a cylindrical superficies.

Corollaries.

Corollaries.

THE superficies of a right cylinder is equal to a rectangle contained under the side (*BC*) *and the circumference of the base.* Fig. 5, 6.

The double of BC (as hath been shew'd above) is to GH, as GH is to BA, or AN; that is, (by the 7th of this) as the circumference P is to the circumference BN. Therefore the triangle under the first, to wit, the double of BC; and the fourth, to wit, the circumference BN, is equal to a triangle under the second GH, and the third, to wit, the circumference P, (as appears from 16. *l.* 6. But the triangle under GH, and the circumference P, is (by the 5th of this) equal to the circle GPH, that is, (by the 11th of this) to the cylindrical superficies. Therefore also the triangle under the double of BC, and the circumference BN, (that is, as appears from 41. *L.* 1. the rectangle which is under BC and the circumference BN) will be equal to the cylindrical superficies. *Q. E. D.*

From this *Corollary* it is manifest, that the properties of rectangles are common to them with cylindrical superficies. Therefore let this be *Corollary* 2. Fig. 20,

2. The cylindrical superficies (BM, QN) which are of 21. *l.* 12.
the same height, are betwixt themselves as the diameters of their bases (BF, QR.)

For the rectangles under the circumference (CL, SE) and the equal right lines FM, RN, to which (by *Coroll.* 1.) the cylindrical superficies are equal, are betwixt themselves (by 1. *l.* 6.) as the bases, to wit, the circumferences CL, SE; that is, as the diameters BF, QR, (by the 7th of this.)

3. The cylindrical superficies (CI, AR) which have equal bases, are betwixt themselves, as their altitudes (TI, BR.) Fig. 23.

For the rectangles contained under the equal circumfer- 24. *l.* 12.
ences GB, MQ, and the sides TI, BR, to which (by *Coroll.* 1.) the cylindrical surfaces are equal, are betwixt themselves (by 1. *l.* 6,) as TI, BR. Fig 20,

4. Like cylindrical surfaces (BM, RI) have betwixt 21. *l.* 12.
themselves a proportion duplicate to that which (BF, QR) the diameters of the bases have. Seeing

Seeing the cylinders are ſuppos'd to be like. M F will be to I Q (by *Defin.* 4. *l.* 12.) as B F to Q R; that is, (by the 7th of this) as the circumference C L to the circumference S E. Wherefore the rectangles alſo which are contain'd under the circumferences C L, S E, and the ſide M F, I Q, will be like; and conſequently they will have betwixt themſelves (by 20. *l.* 6.) a proportion duplicate to that which M F hath to I Q; that is, B F to Q R. Therefore the cylindrical ſurfaces alſo have the ſame.

The ſame Figure. 5. Cylindrical ſurfaces (B M, R I) have betwixt themſelves a proportion compounded of the proportions of the ſides (F M, I Q,) and the diameters of the baſes (B F, Q R,) as is manifeſt from 23. *l.* 6. and the 7th of this.

Fig. 24, 25. *l.* 12. 6. If cylindrical ſurfaces (A R, F D) be equal, as the diameter (A B) is to the diameter F N) ſo reciprocally (by 14. *l.* 6.) the altitude (F H) will be to the altitude (B R) and converſly.

7. Laſtly, from the ſame firſt *Corollary* is had the meaſure of a cylindrical ſuperficies; to wit, if the circumference of the baſe be multiplied by the altitude. As if the altitude be of 20 feet, the circumference of the baſe of 6; multiply 20 by 6, there ariſe 120 ſquare feet for the cylindrical ſuperficies.

PROP. XII. Theorem.

THE ſuperficies of a right cylinder is to the baſe (A B N) as the ſide of the cylinder (C B) is to (B O) the fourth part of the diameter of the baſe.

Let G H be a mean proportional betwixt B C the height, and B D the diameter of the baſe, and conſequently (by *Lemma* before *Prop.* 11. of this) a mean proportional betwixt A N and the double of B C. The circle G P H, of the radius G H, is (by the 11th of this) equal to the curve cylindrical ſuperficies C D. But the circle G P H hath to the baſe of the cylinder A B N a proportion duplicate (by 2. *l.* 12.) to the proportion of G H to A N; that is, the ſame which the double of B C hath to the radius B A (by the hypotheſis, and *Def.* 10. *l.* 5.) that is, the ſame which B C hath to B O, the fourth part of the diameter.

meter. Therefore the cylindrical ſuperficies alſo is to the baſe A B N, as B C is to B O, the fourth part of the diameter. *Q. E. D.*

Corollary.

THE ſuperficies of a cylinder which hath its ſides equal to the diameter of its baſe, is four fold of the baſe. But if the ſide be a fourth part of the diameter of the baſe, the ſuperficies of the cylinder will be equal to the baſe. Both theſe are manifeſt from the propoſition.

PROP. XIII. Theorem.

A Circle, whoſe radius (*OL*) *is a mean proportional betwixt the ſide* (*BC*) *of a right cone, and the radius of the baſe* (*AC*) *is equal to the conical ſuperficies.* *Fig.* 9. 8.

Let regular polygons, E F, I N, be underſtood to be circumſcrib'd about the circles A C G, O P L, and a pyramid circumſcribed about the cone to be erected upon the polygon E F.

Becauſe, by the hypotheſis, A C, or A G. is to O L, as O L is to B C, the proportion of A G to B C, will (*Defin.* 10. *l.* 5.) be duplicate to the proportion of A G to O L. But as A G is to B C, ſo is the triangle under A G, and the circuit E F, to the triangle under B C and the ſame circuit E F. Therefore the proportion of the triangle under A G, and the circuit E F, to the triangle under B C, and the ſame circuit, is alſo duplicate to the proportion of A G to O L. But the triangle under A G and the circuit E F is equal to the polygon L F (by the 4th of this:) And the triangle under B C and the ſame circuit E F (by the 9th of this) is equal to the ſuperficies of the circumſcribed pyramid. Therefore the proportion of the polyon E F, to the ſuperficies of the pyramid, is alſo duplicate to the proportion of A G to O L. But the proportion of the polyon E F to the polygon I N, which is, by the conſtruction, like to it, is (*per* 1. *l.* 12.) alſo duplicate to the proportion of A G to O L. Therefore the polygon

polygon E F hath the ſame proportion to the ſuperficies of the pyramid, and to the polygon I N, which conſequently are equal. In the ſame manner I might ſhew, that the ſuperficies of pyramids, which may be circumſcribed about a cone infinitely more and more, are always equal to polygons which may be circumſcribed infinitely about the circle O P L. Wherefore ſeeing both the ſurfaces of pyramids (by the 10th of this) do at laſt end in the ſurface of the cone and polygons (by the 3d of this) in the circle O P L, the ſuperficies of the cone and the circle O P L, ſhall likewiſe be equal. *Q. E. D.*

From this excellent theorem a circle is found which is equal to a conical ſurface.

Corollaries.

Fig. 9, 8. 1. THE ſuperficies of a right cone, is equal to a triangle comprehended under the ſide of the cone (B C) and the circumference of the baſe (C G.)

Let O L, the radius, be a mean proportional betwixt A C and B C. Then becauſe (by the 7th of this) the circumference C G is to the circumference P, as the radius A G is to the radius O L; that is, by the hypotheſis, as O L is to B C; the triangle under the firſt, to wit, the circumference C G, and under the fourth B C (as appears from 16. *l.* 6.) will be equal to the triangle under the ſecond, to wit, the circumference P, and the third O L; that is, (by the 5th of this) to the circle O P L; that is, to the conical ſuperficies (by the 13th of this) B C D. *Q. E. D.*

From this *Corollary* it appears that conical ſurfaces have the ſame properties with triangles. And ſo it follows.

Fig. 20, 21, *l.* 12. 2. That the conical ſuperficies (B A F, Q X R) having their ſides (B A, Q X) equal, are betwixt themſelves as the diameters of their baſes (B F, Q R.) And,

Fig. 23, 24. *l.* 12. 3. Thoſe which have equal baſes, C F T, A Z B, are betwixt themſelves as their ſides (C F, A Z.) And,

Fig. 20, 21. *l.* 12. 4. Thoſe conical ſuperficies (B A F, Q Z R) which are like, have betwixt themſelves a proportion duplicate to that which is betwixt the diameters of the baſes. And,

The ſame Figure. 5. All conical ſuperficies whatſoever have betwixt themſelves a proportion which is compounded of the proportions of

of the ſides (B A, Q Z) and of the diameters (BF, Q R) which are in the baſes. And,

6. Thoſe which are equal have their ſides and the diameters of the baſes reciprocally proportional; and thoſe which have them ſo, are equal.

All which is demonſtrated from *Corollary* 1. as above we deduced the *Corollaries* concerning the cylindrical ſurface out of the firſt *Corollary* there.

7. Laſtly, we may meaſure a right conical ſurface, if we multiply the ſide F C by half the circumference of the baſe. As if the ſide be of 5 feet, the circumference of the baſe of 20; multiply 5 by 10, and there will ariſe 50 ſquare feet for the conical ſuperficies. The demonſtration is manifeſt from the ſame firſt corollary. *Fig.* 25. *l.* 12.

PROP. XIV. Theorem.

THE ſuperficies of a right cone is to the baſe, as the ſide (BC) is to (AC) the radius of the baſe. *Fig.* 8, 9. *of this.*

Between the ſide B C and A C, the radius of the baſe, let O L be a mean proportional. Therefore the proportion of B C to A C, is duplicate to the proportion of O L to A C (*Defin.* 10. *l.* 5.) Now (by the 13th of this) a circle of the radius O L is equal to the conical ſuperficies C B D. But the proportion of this to A C G, the baſe of the cone; is (by 2. *l.* 12.) duplicate to the proportion of O L to A C; and conſequently is the ſame with the proportion of B C to A C. Therefore the proportion of the conical ſuperficies C B D is to the baſe A C G, as B C is to A C. *Q. E. D.*

Corollaries.

THE ſuperficies of a right cone produced by an equilateral triangle turned about the perpendicular (KA) is double to the baſe (QT.) *Fig* 27.

For the ſide K B is equal to B D, and conſequently double to the half of it A B, which is the radius of the baſe.

2. The

Fig. 24. 2. *The superficies of a cone produced by a right-angled equicrural triangle (EBD) is to the base, as in a square the diameter is to the side.*

For the perpendicular B A being drawn, the right angle B (by 26. *l.* 1.) is bisected, and consequently A B D is half right. But A D B is also an half right angle (by *Corol.* 11. *p.* 32. *l.* 1. Therefore D A, B A, are (by 6. *l.* 1.) equal; and consequently B D is the diameter of the square A K, whereof A D is the side. Now the same A D is the semi-diameter of the base P T, seeing the perpendicular A B (by 26. *l.* 1.) bisects E D. From which, and this fourteenth, the corollary is manifest.

Fig. 24. 3. *The superficies of the right cylinder (GK) is to the superficies of the right cone (GBN,) as the side of the cylinder is to half the side of the cone.*

For the superficies of the cone, G B N, is to the base M I, as the side B N is to Q N, the semi-diameter of the base (by the 14th of this) that is, as half the side B N is to the fourth part of the diameter G N. But the base M I (by the 12th of this) is to the superficies of the cylinder G K, as the fourth part of the diameter is to N K, the side of the cylinder. By equality of proportion therefore the conical superficies G B N is to the cylindrical superficies G K, as half the side of the cone is to N K, the side of the cylinder. *Q. E. D.*

A lemma to what follows.

Fig. 10. IN a triangle, as N P V, let there be drawn Q D parallel to N V.

I say, that the rectangle under P N and N V is equal to the rectangle under P Q, Q D, together with the rectangle under N Q, and the two N V, Q D put together.

Draw N A perpendicular to the side N P, and equal to N V; and the rectangle N O being compleated, let the diameter P A be drawn. Then from Q let there be drawn

Q E

QE parallel to N A, which may cut P A in B. Through B let C F be drawn parallel to N P. Becauſe A N is equal to N V, it is manifeſt that Q B alſo is equal to Q D.(from *Corollary* 1. *p* 4 *l*. 6.) Therefore the Rectangle O N is the Rectangle P N V, and F Q is P Q D. It remains, that we prove that the Rectangles OB, EC, BN. are equal to the Rectangle under N Q, and the two N A, B Q. that is, to the Rectangle under N Q; and the two Lines N V, Q D. But that is manifeſt; for the Rectangle under N Q and N A, Q B, is equal (*per* 1. *l*. 2.) to theſe three Rectangles, that under N Q and C A (that is, the ſpace E C) and that under N Q and N C (that is, the ſpace B N) and that under N Q and Q B, that is again the ſpace B N, and conſequently the ſpace O B, which (*per* 43 *l*. 1.) is equal to B N. The Propoſition therefore is manifeſt.

PROP. XV. Theorem.

IF a right Cone be cut by the Plane QSB, parallel to N Z O; I ſay, that the Circle G H M whoſe Radius GH is a Mean betwixt Part of the Side N Q and Q D, NV taken together, the Radius's of the Circles QSB, NZO, is equal to the conical Surface intercepted betwixt the parallel Circles Q S B, N Z O. Fig. 11. 12

Let G F be the Mean betwixt P N and N V. Likewiſe let GK be the Mean betwixt P Q and Q D; and let there be deſcribed the Circles GFL, GKT. This (by the 13th of this) will be equal to the conic Superficies Q P B, and the other to the Superficies N O P. The Rectangle PNV, (by the *Lemma*) is equal to the Rectangle P Q D, together with the Rectangle under N Q and N V Q D, taken together. But becauſe (by the Conſtruction) G F is a mean Proportional betwixt P N, N V; the Rectangle P N V is equal to the ſquare of G F (by 17. *l*. 6.) And becauſe G K is (by the Conſtruction) a Mean betwixt P Q, Q D, the Rectangle (by 17. *l* 6.) P Q D is equal to the ſquare of G K; And becauſe G H, by the Hypotheſis, is a Mean betwixt Q N and Q D, N V, taken together, the Rectangle (by 17. *l*. 6) under Q N, and Q D, N V. taken together,

P is

is equal to the ſquare of G H. Therefore the ſquare of G F is alſo equal to the ſquare of G H, and to that of G K. Therefore ſeeing Circles are betwixt themſelves (by 2. *l.* 12.) as the ſquare of the Radius's, the circle G L F will alſo be equal to the two circles G K T, G H M taken together. But (by the 13th of this) the circle G L F is equal to the conical ſuperficies NPO. Therefore the conical ſuperficies NPO is alſo equal to the two circles G K T. and GHM. But Q P R, one Part of the ſuperficies N P O, is (by the ſame) equal to the circle G K T Therefore the remaining Part, which is comprehended betwixt the parallel circles ZZ, SS, is equal to the circle G H M. *Q. E. D.*

A Lemma to what follows.

Fig. 13. RIGHT Lines (BH, CG) which in the circle intercept equal Arches (B C, H G) are parallel.

For let CH be drawn. Becauſe the Arches BC, HG, are by the Hypotheſis, equal; the Alternate Angles alſo (by 29. *l.* 3.) B H C, G C H, will be equal. Therefore (by 28. *l.* 1) BH, and CG, are parallel. *Q E D.*

PROP. XVI. Theorem.

Fig. 13. *LET there be inſcribed in a Circle a regular Figure of an even Number of Sides; and let it be equilateral; let EB be drawn from the Extremity of the Diameter unto B, the end of the ſide next to the Diameter: and let the right Lines BH, CG, DF, join the Angles which are equally diſtant from A.*

I ſay, that the Rectangle contained under the Diameter AE, and the Subtenſe EB, is equal to the Rectangle of one Side of the inſcribed Figure AB, or BC, &c. and of all the joining Lines BH, CG, DF, taken together.

Draw CH, DG: becauſe BH, CG, DF, intercept (by 26. *l.* 3.) equal Arches, B C, H G; C D, G F, theſe Lines

Lines (by the *Lemma*) will be parallel. By the ſame Argument BA, CH, DG, EF, are parallel. All the Triangles therefore (by 27, and 15. *l.* 1.) BAK, KHL, LCM, MGN, NDO, OFE, are Equi-angular. Therefore (by 4. *l.* 6.) as BK is to KA, ſo is HK to KL; and as HK is to KL, ſo is CM to ML; and as CM is to ML, ſo is GM to MN; and as GM is to MN, ſo is DO to ON, and as DO is to ON, ſo is FO to OE. Therefore (by 12. *l.* 5.) as one of the Antecedents BK, is to one of the Conſequents KA; ſo all the Antecedents BK, KH, CM, MG, DO, OF, (that is, all the joining Lines BH, CG, DF) are to all the Conſequents AK KL, LM, MN, NO, OE, (that is, to the Diameter AE.) But (by 8. *l.* 6.) as BK is to AK, ſo is EB to BA. Therefore as all theſe together BH, CG, DF, are to AE, ſo is EB to BA. Therefore (by 16. *l.* 6.) the Rectangle under BA on one Part, and all the joining Lines BH, CG, DF, on the other, is equal to the Rectangle which is under AE and EB. *Q.E.D.*

PROP. XVII. Theorem.

LET there be inſcribed in DAF a Segment of a Circle, whoſe Baſe DF is perpendicular to the Diameter AOE, a Figure equilateral, and of an even Number of Sides; and let there be drawn, as in the foregoing, the Line EB. Fig. 14.

I ſay, that the Rectangle comprehended under EB and AO, part of the Diameter, which is the Axis of the Segment; is equal to the Rectangle, which is under one Side of the inſcribed Figure, and all the joining Lines BH, CG, &c. taken together with DO half the Baſe DF.

The Demonſtration is the ſame with that of the foregoing.

Lemma 1. *to what follows.*

LET there be inſcribed in the greateſt Circle of a Sphere a regular Figure, which hath its Sides meaſured by the number Four, and ſtands about the Axis AE; which Axis remaining unmov'd, let the Circle be turn'd round together with the Figure. Fig. 13.

 I ſay;

I say, that there will be inscribed in the Sphere a Body contained under conical Superficies.

It is manifest (see *Defin.* 2. *l.* 12.) that B A, H A, likewise D E, E F, describe entire Superficies of right Cones. Then, because the Lines CB, G H, and G F, C D, being produced, do concur on both sides in the same Point of the Diameter A E, which is in like manner to be drawn out, and cuts the joining Lines perpendicularly; It is also manifest that the said Lines C B, G H, *&c.* do describe Parts of right conical Surfaces which are intercepted betwixt the parallel Circles, which the Tops of the Angles B, C, D, design in the Spherical Superficies.

Lemma 2.

LET D A F be the greatest Section of a Segment of a Sphere whose Axis is A O. Let there be inscribed in this a Figure having all the sides equal, the Base excepted, and let it be turned round about the Axis AO.

I say, that a Body contained under conical Superficies will be inscribed in the Spherical Segment.

This is proved as the foregoing *Lemma*.

PROP. XVIII. Theorem.

Fig. 13. *LET the same things be supposed which were in the first Lemma; and let the right Line (EB) be drawn from the Extremity of the Diameter unto the End of the Side next to the Diameter.*

I say, that a Circle, the Square of whose Radius (I) is equal to the Rectangle A E B, contained under the Diameter AE, and the Subtense EB is equal to all the conical Surfaces inscribed in the Sphere.

That is a Circle whose Radius (I) is a mean Proportional betwixt A E and E B.

Because the right Lines BH, CG, DF, are equal to the right Lines B K, C M, D O, taken twice, (by 1. *l.* 2) the Rectangle under one side of the figure inscribed in the greatest Circle, (to wit under A B, or B C, or C D, or D E) and under all the joining Lines together, B H, C G, DF, is equal to the Rectangle under AB and BK, together with that which is under BC and the Compound of BK and C M,

CM together with that which is under CD and the Compound of CM and DO, together with that which is under DE and DO; for ſo each of the Lines BK, CM and DO, are taken twice. But (by the 16th of this) the Rectangle under AB and all the joining Lines BH, CG, DF, taken together, is equal to the Rectangle AEB; that is, by the Hypotheſis, to the Square of I. Therefore the Square of I is equal to the Rectangles under AB and BK, and under BC and the Compound of BK and CM, under CD and the Compound of CM and DO, and under DE and DO. Now let P be a mean Proportional betwixt AB and BK, and Q a Mean betwixt BC and the Compound of BK and CM; and R a Mean betwixt CD and the Compound of CM, DO; S, a Mean betwixt DE and DO. The Squares thereof of P, Q, R, S, (by 17. *l.* 6) are equal to the above ſaid Rectangles. Wherefore ſeeing I have already ſhewed the Square of I to be equal to the ſame Rectangles, it muſt alſo be equal to the Squares of P, Q, R, S, together. Seeing therefore (by 2. *l* 12) Circles are betwixt themſelves as the Squares of their Radius's; the Circle deſcribed of the Radius I, will alſo be equal to all the Circles together whoſe Radius's are P, Q, R, S, as is manifeſt from 22. *l.* 6. But the Circles of the Radius's P and S, are by the 13th of this) equal to the conical Superficies which the Sides AB, ED, have produced; foraſmuch as P is a mean Proportional betwixt AB the Side of the Cone, and BK the Radius of the Baſe; and S is a mean Proportional betwixt ED and DO; and the Circle of the Radius Q is (by the 15th of this) equal to that Segment of a conical Superficies which is intercepted betwixt the two parallel Circles of the Diameters CG, BH, becauſe Q is a Mean betwixt BC and the Compound of BK, CM: And for the ſame Cauſe the Circle of the Radius R is equal to a Segment of a conical Surface, which is intercepted betwixt the parallel Circles of the Diameters CG, DF. Therefore the Circle deſcribed from the Radius I, is equal to the conical Surfaces inſcribed in the Sphere taken all together. *Q. E. D.*

PROP. XIX. Theorem.

LET the ſame things be ſuppoſed which were in the ſecond Lemma, and let the right Line EB be drawn from the Extremity of the Diameter (AE) to the End of AB, the Side next to the Diameter.

 I ſay

I say, that a Circle, whose Radius is a mean Proportional betwixt (EB) and (AO) the Axis of the Segment, is equal to all the conical Superficies inscribed in the spherical Segment. D A F.

The Demonstration is altogether the same with that of the foregoing; only for *Prop.* 16. let *Prop.* 17. be cited.

PROP. XX. Theorem.

Fig. 15. *CONICAL Superficies inscribed in a Sphere, do at length end in the Superficies of the Sphere.*

Let there be given a Superficies as small as you will, as X. It is manifest that within the spherical Superficies ACEG, there may be given some other Concentrical thereto, which falls short of this by a Quantity less than X. Let ACEG, D P L M, be the greatest Circle of both, as cut with a Plane through the Centre. Let there be drawn the Diameter A D E, and in D let N Q touch it. If the Arch A E be bisected in C, and again the remainder be bisected, and so on, there will be left at last the Arch AB (as is manifest of itself) less than the Arch AN. If to this Arch the right Line AB be subtended, it is manifest, that it will not reach to the Circumference PDML, and that it will be a Side of an Equilateral Figure of an even number of Sides inscribed in the Circle CAGE, no Side whereof reacheth unto the Circumference PDML. Wherefore if all be turn'd round about the Diameter AE; it is manifest that there will be inscribed in the exterior spherical Surface, conical Surfaces, which include the spherical Surface, which is concentrical to the other, and consequently, by *Axiom.* 3. of this, are greater. Because therefore the spherical Surface DPLM falls short of the spherical Surface A C E G, by a Quantity less than the given one X; much more will the conical Surfaces fall short of the said spherical Surface ACEG by a Quantity less than the given one X and (by *Defin.* 6. *l.* 12.) consequently will end in the Superficies A C E G. *Q E D.*

PROP. XXI. Theorem.

Fig. 17. *CONICAL Superficies inscribed in a spherical Segment DAF, end in the spherical Superficies of the Segment itself.*

This may be demonftrated by the fame Reafoning as the foregoing was.

PROP. XXII. Theorem.

IT was demonftrated, Prop. 18. *that a Circle whofe Radius is a mean Proportional betwixt the Diameter AE, and the right Line EB, which is drawn from the Extremity of the Diameter unto the End of the Side AB, next to the Diameter, is equal to all the conic Superficies infcribed in the Sphere.* Fig. 16.

I fay, that this Circle (fee Def. 6. *l.* 12.) *ends at length in a Circle, whofe Radius is AE, the Diameter of the Sphere.*

For if more and more fides be infinitely infcribed in the greateft Circle (which then being turn'd round about AE, produce conical fuperficies) it is manifeft, that the fide AB becomes at length lefs than any given right Line, and confequently that the fubtenfe EB approaches to the Diameter AE within a Diftance lefs alfo than any given one; from whence it comes to pafs that the Difference of thefe AE, BE, becomes likewife lefs than any given one. Therefore much more fhall the mean Proportional betwixt AE, BE, which is always greater than BE, differ from AE at length by a Defect lefs than any given one Therefore the circle alfo whofe femi-diameter is a Mean betwixt AE and BE, will at length differ from a circle whofe femi-diameter is AE, by a Defect lefs than any given one whatfoever, that is, will end (*Def* 6. l. 12.) in it. *Q.E.D.*

This which is clear enough of itfelf, there is no need to demonftrate more operofely.

PROP. XXIII. Theorem.

IT was demonftrated, Prop. 19. *that a Circle, whofe Radius is a mean Proportional betwixt EB and the Axis of the Segment AO, is, equal to all the conical Superficies infcribed in the fpherical Portion DAF.* Fig. 17.

I fay

I say, that this Circle ends in a Circle, whose Radius is the right Line AD, drawn from the Vertex of the Segment unto the Circumference of the Circle DQFN, which is the Base of the Segment.

For because it now appears from the foregoing Demonstration that EB doth at length end in AE; it will also be manifest that the mean Proportional betwixt EB and AO doth at length end in the mean Proportional betwixt AE and AO, that is, (by *Coroll* 2. *p* 8. l. 6.) in AD itself. It is therefore manifest that the Circle also whose Radius is a mean Proportional betwixt EB and AO, doth end in the Circle of the Radius AD. *Q. E. D.*

A Lemma to the following Proposition.

IF the Diameter of one Circle be double to the Diameter of another, the one Circle will be four-fold to the other. This is manifest from *Prop.* 2. *l.* 12. and *Defin.* 10. *l.* 5.

PROP. XXIV. Theorem.

Fig. 16. *THE Superficies of every Sphere is four-fold of the greatest Circle of the same Sphere.*

This most noble Theorem of *Archimedes* we shall, from what goes before, expeditiously demonstrate in this manner.

Let an Ordinate Figure, the Sides whereof are measured by the number Four, be understood to be inscribed in the greatest Circle of a Sphere about the Diameter AE. Let this Figure be turned round about AE, and so produce conical Surfaces inscribed in the spherical Surface, and let EB be drawn. It hath been demonstrated above (18 of this) that all conical Surfaces inscribed in a sphere are equal to the Circle, the Square of the Radius whereof is equal to the Rectangle AEB, that is, whose Radius is a mean Proportional betwixt AE and EB. And this will always happen although the Inscription be infinitely continued. Wherefore seeing the inscribed conical surfaces (by 20 of this) will at length end in the spherical surface, and the Circle whose Radius is a Mean betwixt AE and EB, will at length end (by 22. of this) in the Circle whose Radius is AE; the spherical surface itself also (by 2. of this) will be equal

to

to the Circle of the Radius AE, that is, by the foregoing *Lemma*, to four times the greateſt Circle ACEG. *Q. E D.*

He that ſhall read *Archimedes*, will find that the Way here uſed in demonſtrating this moſt noble Theorem, is much ſhorter and clearer than that of *Archimedes*.

Corollary.

FROM this admirable Theorem, whereby *Archimedes* hath purchas'd to himſelf an immortal Name amongſt the Geometricians, a Circle equal to a ſpherical Surface is obtained; that, to wit, whoſe Semi diameter is the Diameter of a Sphere, or whoſe Diameter is double to the Sphere's Diameter.

Scholium.

WE are now well provided for the meaſuring of a ſpherical Surface, the chief amongſt all Curve ones. And it is performed theſe two Ways.

1. Let the greateſt circle of the ſphere be meaſured (according to *Schol. Prop.* 6 of this) and let it be multiplied by 4. As, if the greateſt circle of the Orb of the Earth be found to contain 49,075,500 ſquare Miles, or more exactly, 49,081,250 ſquare Miles, then, according to this 196,325,000 ſquare Miles are contain'd in the whole ſpherical Surface.

2. The Diameter of a Sphere multiplied by the circumference of the greateſt circle, gives you the ſpherical Superficies. According to which, if the Earth's Diameter conſiſt of 7,853 Miles, and conſequently the circumference of the greateſt circle conſiſts of 25,000, the whole ſpherical Surface will be in the ſame Miles 196,325,000; for 7,853 ×25,000=196,325,000.

The Demonſtration appears from *Corol.* 1. *Prop.* 5. of this; for a Rectangle under the Diameter of a Sphere, and the circumference of the greateſt circle, is according to that *Corollary* four-fold of the greateſt circle.

PROP. XXV. Theorem.

THE Surface of any ſpherical Portion whatever (as DAF) is equal to a Circle, whoſe Radius is the right Line (AD) drawn from the Vertex of , *Fig* 17,

the

the Portion to the Circumference of the Circle (DQFN) which is the Basis of the Segment.

Let a Figure, Equilateral and of an even Number of sides, the base being set aside, be imagin'd to be inscrib'd in the section of the greatest Portion about the Axis AO; this Figure being turn'd round about AO, will inscribe conical surfaces in the Portion. Let the right Line EB be drawn also as above (in 18 and 19 of this) All the conical Surfaces now inscribed are equal (by the 19 of this) to the Circle whose Radius is a mean Proportional betwixt EB and the Axis of the segment AO. And this will always happen if the Inscription be infinitely continued. Wherefore seeing both the conical surfaces inscrib'd in the segment end at length (by 21 of this) in the spherical Surface of the Segment and the circle whose Radius is a Mean betwixt EB and AO, ends (by 23.) in the circle of the Radius AD; the spherical surface of the Portion also DAF (by 2) will be equal to the circle of the Radius AD. *Q.E.D.*

This is another of the more noble Inventions of *Archimedes*, which, as the former, we have demonstrated in a much shorter and clearer Way, than he did.

PROP. XXVI. Theorem.

Fig. 18. *THE Superficies of a right Cylinder circumscribed about the Sphere (as the Cylinder HPSV) is equal to the Surface of the Sphere.*

And if a Cylinder and Sphere be cut by Planes perpendicular to the Axis (BG;) each Segment of the cylindrical Surface will be equal to each Segment of the spherical Surface.

Part I. Because the side HP of the cylinder is (by the Hypothesis) equal to PS, the Diameter of the base; the cylindrical surface HS will be (by *Coroll. p.* 12 of this) four fold of the base; that is, of the greatest circle of the sphere inscrib'd in the Cylinder; of which, seeing [by the 24 of this] the spherical Surface itself is also four fold, this will be equal to the cylindrical Surface. *Q. E. D.*

Part II. Let the right Lines BO, GO, be drawn. Because the Angle BOG (by 31. *l* 3.) is right, as being the Angle

Angle in the Semi-circle, and O C falls perpendicular from it upon B G; B O (by *Coroll.* 2. *p.* 8. *l.* 6.) will be a mean Proportional betwixt G B and B C, that is, betwixt I T and H I. Therefore the Circle of the Radius B O (by 11. of this) will be equal to the cylindrical Surface H T. but the same Circle is also (by the foregoing) equal to the Segment of the spherical Surface O B K. Therefore the cylindrical surface H T, and the spherical OBK are equal.

Then because it is shew'd in the same manner that the cylindrical surface H X is equal to the spherical Q B R, the remaining cylindrical surface I X will be equal to the remaining spherical Surface Q O K R, which is intercepted betwixt two parallel Circles.

And from these the Proposition is manifest of all segments.

[Corollary. "Hence the superficies of a Cylinder circumscrib'd about a circle is double to the base's."]

PROP. XXVII. Theorem.

THE Segments of a spherical Surface divided by parallel Circles have that Proportion amongst themselves, which the Segments (BC, CD, DA, AE, EF, FG,) of that Diameter (BG) which is perpendicular to the parallel Circles have amongst themselves. Fig. 18.

It follows from the foregoing. For by that the segments of the spherical surface O B K, Q O K R, M Q R N, &c. are equal to the cylindrical H T, I X, L N, &c. But these (by 13. *l.* 12) have the same Proportion betwixt themselves, which the segments of the Axis B C, C D, D A, &c. have. Therefore those also have the same Proportion. *Q E D.*

Scholium.

FROM hence the Proportion of Zones and Climates betwixt themselves becomes known. For they are to one another as the segments of the Axis, which are known from the Table of Sines. Fig. 19.

From the same also we learn to measure the Segments of a spherical Surface. For because both the whole Surface of the Sphere is known from *Schol. Prop.* 24. and the Proportion

tion of the ſegments, the ſame as that of the Parts of the Axis; is alſo given; it is manifeſt that each of the ſegments become known.

Now both the foregoing, and all the reſt of the Theorems which follow, are altogether ſingular and admirable, and well worthy that thoſe who are ſtudious of Geometry ſhould give all Diligence to underſtand them.

A Lemma to the following.

Fig. 19. IF a Plane (QN) touch a Sphere in (O) a right Line (AO) from the center to the contact, is perpendicular to the Plane.

Let Q N, the touching Plane and the Sphere, be cut through the Contact with two Planes, which in the Sphere may produce the circles O G, O D, but in the Plane Q N, the right Lines C O, I O, which ſhall touch the circles in O. Therefore by 18. *l.* 3. A O is perpendicular to both I O and C O, and conſequently by 4 *l.* 11. perpendicular to the Plane Q N. *Q. E D.*

PROP. XXVIII. Theorem.

Fig. 20, 22, 21. *EVERY Sphere is equal to a Cone (ZO) whoſe Altitude (KO) is equal to the Radius of the Sphere; and the Baſe (Z) equal to the Superficies of the Sphere.*

Let ſome Polyedral Body be underſtood to be circumſcribed about the Sphere, and let the ſolid Angles thereof be cut off by new Planes touching the Sphere. Which being done, there will ariſe another Polyedral Body containing the Sphere, but leſs than the former, and conſiſting of more Angles, and having a ſurface compounded of more Tangent Planes in Number, but leſs in Magnitude. If the ſolid Angle of this Polyedrum be again cut off by new Tangent Planes, and the Angles of the third Polyedrum thence ariſing likewiſe, and ſo on for ever; it will come to paſs at length, that both the Polyedrum will exceed the ſphere by a ſolid leſs than any given one whatſoever; and the ſurface thereof compounded of Tangent Planes (which, as I ſaid, are endleſly leſs in Magnitude, and more and more in Number than they were before) will exceed the ſpherical ſurface alſo by a

Plane

Plane less than any given one whatever. Both which things although they might be demonstrated, yet because they are of themselves manifest enough, I shall for Brevity sake take for granted. These things being thus stated, we proceed.

The Polyedrum now defin'd is compounded of Pyramids; the common Top whereof is the center of the sphere, and the Bases are Tangent Planes, which constitute the surface of the Polyedrum. And because the right Lines drawn from the center A unto the contacts of each of the Planes, are (by the foregoing *Lemma*) perpendicular to each of the Planes: therefore the Height of all the Pyramids, whereof the Polyedrum consists, will be equal; to wit, A B the Radius of the sphere. If therefore the Plane X be supposed equal to the surface of the Polyedrum itself, and upon it there be erected a Pyramid at the Height M N, which is also equal to the Radius of the sphere A B; it is manifest (by 6. *l.* 12) that all the abovesaid Pyramids, that is, the whole Polyedrum are equal to the Pyramid X N After the same manner all the rest of the Polyedrums containing the sphere, which from the perpetual abscission of the solid Angles will arise one after another infinitely, are always equal to the Pyramids (represented by XN) the Altitudes whereof M N are the Radius of the sphere; but the Bases X, equal to the surfaces of Polyedrums encompassing the sphere. Wherefore seeing at length both the Polyedrums (as I said above) do end in a sphere, and the Pyramids (X N) as I will shew by and by, do end in the Cone Z O; the sphere also (by 1. of this) will be equal to the Cone. *Q.E.D.*

But that the Pyramids X N end in a cone, I thus shew: The surfaces of Polyedrums end in the surface of the sphere as it was taken for granted above. But the Bases X of the Pyramids X N, are always suppos'd equal to the surfaces of the Polyedrums; and Z the Base of the Cone, Z O is, by the Hypothesis, equal to the surface of the sphere; therefore the Bases X also will end in the Base Z; and consequently seeing the Pyramids X N to be to the Cone which, by the Hypothesis, is of equal Height (by *Coroll. Prop.* 11. *l.* 12.) as the Base X is to the Base Z, the Pyramids also will end in the Cone.

The Demonstration of this Proposition and the following is altogether diverse from that which *Archimedes* made use of which indeed is very subtil and ingenious, but prolix and difficult

difficult; to which there are premised two Positions that are manifest, and eleven Propositions, besides others, not a few, on which they depend. But the Theorem itself, as propounded by *Archimedes*, is thus: Every Sphere is fourfold of a cone, which hath a base equal to the greatest circle of the Sphere, and its Altitude equal to the Radius.

Scholium.

FROM this noble Theorem is deduced the Mensuration of the most noble of solid Figures. For if the sixth Part of the Diameter, or the third Part of the Semidiameter be multiplied by the Surface of the Sphere, already known by *Schol. Prop.* 24, there will arise the Solidity of the Sphere.

Suppose the superficies of the Earth be found to contain 196,325,000 square Miles, and let the third Part of the Semi-diameter consist of 1309 such Miles. Multiply the two Numbers together, the Product 256,989,425,000 will be the Number of the cubic Miles of the Earth's solidity.

For seeing a Sphere (by this *Prop.*) is equal to a Cone whose Altitude is the Radius of the Sphere, and its Base the Surface of the same Sphere, and the Solidity of the Cone (by *Schol. Prop.* 6. of this) is produced from the third Part of the Altitude (that is, the Radius of the Sphere) multiplied by the Base (that is, the Surface of the Sphere) the Sphere's Solidity also is obtain'd from the third Part of the Radius multiplied into the Superficies.

PROP. XXIX. Theorem.

Fig. 23. *EVERY Sector of the Sphere is equal to a Cone whose Altitude is the Radius of the Sphere, and the Base the spherical Superficies of the Sector.*

First, let the sector AECG be less than an Hemisphere. Let a right-lin'd polyedral Body be understood to be circumscrib'd about the Sector. Now, if all the remaining Ratiocination be carried on after the same manner as was done in the foregoing, the Thing sought will be concluded in the same manner. This Thing alone will require to be shew'd upon which indeed the whole Reasoning depends; to wit, that the Superficies of the Polyedrum, which is compounded of

of Planes on every Side, touching the Surface of the Sphere ECG, is greater than the Surface ECG. Which is done thus. Let another equal and like Surface be conceiv'd to be set to the Surface ECG encompass'd with touching Planes in the very same manner as the other is. There will now (by *Axiom* 3. of this) the whole Surface compounded of Planes, be greater than the whole spherical Surface. Therefore half the Surface compounded of Planes will also be greater than half the spherical Surface E C G.

Then let the Sector AEBG be greater than an Hemisphere. Both Sectors taken together, are (by the foregoing) equal to a Cone whose Height is the Radius of the Sphere, and its Bases the whole Superficies; that is, (by 11 *l.* 12) to two Cones, which have the same Height, but have their Bases equal to the Segments of the spherical Superficies E C G, E B G. But one of the Sectors A E C G, that which is less than an Hemisphere is by Part I. equal to a Cone, whose Altitude is the Radius, and its Base the Surface ECG. Therefore the other Sector EABG, is equal to the other Cone, whose Height is the Radius, and its Base the remaining spherical Surface EBG. *Q.E.D.*

Corollary.

SEEING (by 25 of this) the Superficies E C G is equal to the Circle of the Radius C G, and the Superficies EBG equal to the Circle of the Radius BG, the Sectors A E C G, and A E B G, will be equal to Cones, whose Altitude is the Radius of the Sphere, and their Bases, Circles of the Radius's CG and BG.

Scholium.

FROM these Things is deduced the measuring both of Sectors and Segments of Spheres: of Sectors (as appears from *Schol. Pr*[illegible]. of this) if the third Part of the Radius be multiplied by the spherical Surface of the Sectors, which is already known from *Schol. Prop.* 27. or by the Circle of the Radius CG, or BG: and of Segments, if the Cone E A G be measured, and be taken away from the Sector; if it be less than an Hemisphere, but added thereto, if it be greater. *Fig.* 23.

The

Fig. 18. The Segment (MQRN) which lies betwixt two Circles, whether parallel or not parallel, is meafured; if the Segments QBR and MBN already known, be fubftracted one out of the other.

PROP. XXX. Theorem.

Fig. 24. *AN Hemifphere (E O B D) is double to the Cone (E B D) which hath the fame Bafe and Altitude with itfelf.*

The Cone, whofe Bafis is the Hemifpherical Superficies, E O B D, and its Altitude the Radius A B, is to the cone E B D (by 11 *l.* 12) as bafe is to bafe, that is, as the hemifpherical Surface E O B D, is to the greateft circle P T. Therefore feeing the hemifpherical Superficies E O B D is double to the greateft circle (by 24 of this) the cone alfo which hath the Superficies E O B D for its bafe, and the Radius A B for its Altitude, is double to the cone E B D. But (by 28 of this) the Hemifphere is equal to a cone which hath the Radius for its Altitude, and the hemifpherical Superficies for its bafe. Therefore the Hemifphere is alfo double to the cone E B D. *Q E D.*

PROP. XXXI. Theorem.

Fig. 25. *LET a Sphere be divided into two Segments I L, B G, I S, K G, by the Plane, IQ, GT, which doth not pafs through the Center A; and let the Diameter B O K be perpendicular to the cutting Plane.*

As the Altitude O B of the Segment, I L B G, is to the Radius of the Sphere A B; fo let O K, the Altitude of the other Segment be made to the other Line K N.

In like manner, as O K, the Altitude of the Segment I S K G, is to the Radius A K or A B fo let the Altitude O B of the other Segment be made to the other Line B D. Which Things being fuppofed, I fay.

1. *The*

1. The cones I N G and I D G whose altitudes are O N, O D, and I Q G T, their common base, are equal to the spherical segments.

2. There is the same proportion of the segments as there is of the right lines D O, N O.

3. The segment I S K G is to the greatest cone I K G inscribed in it, as N O is to K O; and the segment I L B G is to the greatest cone I B G inscrib'd in it, as D O is to B O.

Part I. Let the sphere and cones be cut by a plane through the diameter B K. There will be produced in the sphere the greatest circle B L K G and in the cones the triangles B I G, I K G. And because B O K, the diameter is (by the hypothesis) perpendicular to the circle Q T, I O B (by *Def.* 3. *l.* 11.) will be a right angle. The angle in the semi-circle is also a right one (by 31. *l.* 3.) Because therefore in the triangle B I K, there is drawn from the right angle, I O perpendicular to the base B K; B I will be to I O, as (by 8. *l.* 6.) B K to K I. Therefore the duplicate proportion of B I to I O is equal to the duplicate proportion of B K to K I; that is, (because B K, K I, K O by *Coroll.* 2. *p.* 8. *l.* 6. are three proportionals) equal to the proportion of B K to K O.

Then because O B is (by the hypothesis) to B D, as O K is to the radius A B; by inversion it will be always thus, D B is to B O, as AB to O K; and by permutation thus, D B is to B A, as B O to O K; and by compounding thus, D A is to B A, as B K is to O K. Because therefore I have already shew'd the proportion of B K to O K, to be duplicate to the proportion of B I to I O, and consequently (by 2. *l.* 12.) equal to the proportion betwixt the circles describ'd by the radius's B I, I O; D A will also be to B A, as the circle of the radius B I, to the circle of the radius I O. Therefore the cone under the altitude D A, and for the base the circle of the radius I O; that is, the circle Q T, is equal to the cone under the altitude B A (by 15. *l.* 12.) which hath for its base the circle of the radius B I; that is, (by *Coroll.* 29. of this) the spherical sector A I B G. Wherefore if the same cone I A G be added, as well to the sector A I B G, as to the cone under D A, and the circle Q T, the wholes will be equal, to wit,

 the

the ſpherical ſegment I L B G, will be equal to two cones, whereof one is that which is under the baſe Q T and the altitude D A, and the other I A G is under the ſame baſe Q T, and the altitude O A. But theſe two cones (by 14. *l.* 12.) make up the cone I D G. Therefore the ſegment I L B G will be equal to the cone I D G. *Q. E. D.*

By the ſame reaſoning, the ſegment I S K G will be equal to the cone I N G, with this only change, that the cone I A G, which before was added, be now taken away.

Part II. This is manifeſt out of the firſt. For the cones I D G and I N G are betwixt themſelves (by *p.* 14. *l.* 12.) as are D O and N O. Therefore the ſegments alſo I L B G, I S K G, equal to thoſe cones, are betwixt themſelves, as the right lines D O, N O.

Part III. This likewiſe is manifeſt from the firſt. For the cone I D G is to the cone I B G, (by the ſame) as D O is to B O. Therefore the ſegment alſo I L B G, which is equal to the cone I D G, is to the cone I B G, as D O is to B O.

Scholium.

FROM the firſt part of this propoſition there ariſes another way of meaſuring ſpherical ſegments, and that a very eaſy one; if, to wit, the cones I D G, I N G, be meaſured; which will be done, if the third parts of the right lines D O, N O be drawn into the circle Q T.

PROP. XXXII. Theorem.

Fig. 24. *A Right cylinder (G K) is both in ſolidity and the whole ſuperficies to the ſphere about which it is circumſcrib'd, as* 3 *to* 2.

Let B Q be the common axis of the ſphere and cylinder, and E B D the greateſt cone inſcrib'd in the hemiſphere E O B D. Becauſe the cylinder E K (half of G K is (by 10. *l.* 12.) triple to the cone E B D, while the hemiſphere is double to the ſame cone, (by 30. of this) it is manifeſt that the cylinder E K is to the hemiſphere as 3 to 2. Therefore alſo the whole cylinder G K, is to the whole ſphere Q E B D, as 3 to 2. Which was the firſt part.

Then becauſe the ſide of the cylinder K N is equal to G N the diameter of the baſe, its ſuperficies without the baſes will be four-fold (by *Coroll. p.* 12. of this) of the baſe

base M I, and consequently taken together with the bases, that is, the whole superficies of the cylinder, will be sixfold of the base M I, which is equal to the greatest circle of the sphere. But the superficies of the sphere is fourfold of that greatest circle. Therefore the whole superficies of the cylinder G K is to the superficies of the sphere, as 6 to 4, or as 3 to 2. Which was the other part.

Therefore a cylinder is both in solidity and the whole superficies to the sphere, about which it is circumscrib'd, as 3 to 2. *Q. E. D.*

Scholium.

IT is an argument what a great value *Archimedes* puts upon this theorem, that he would have a sphere inscrib'd in a cylinder set upon his tomb. And perhaps amongst so many other famous discoveries, this chiefly and above all others pleas'd him for this reason, to wit, because there was one and the same rational proportion both of bodies, and of the surfaces which contain them. We have demonstrated a like identity of affections betwixt rings, and the surfaces of rings, in the fourth book of our cylindricks and annularies, *Prop.* 13, 14, 15. And another famous example of the same hath also offer'd itself to me in the sphere itself. For I have found, that like as a sphere is to a right cylinder which encompasseth it (which will necessarily be equilateral) as 2 is to 3, and this both in respect of solidity and surface; so likewise the sphere hath to an equilateral cone encompassing it, that proportion which 4 hath to 9; and this both in regard of solidity and superficies. From which this also follows, That the sesquialteral proportion found by *Archimedes* in the sphere and cylinder is contained in three solids, sphere, cylinder and equilateral cone. The demonstration of both which things, with some other theorems of my own, in which the wonderful nature of the sphere will more appear, I shall subjoin in the thirteen following propositions.

PROP. XXXIII. Theorem.

THE superficies of a sphere is double to the superficies of a square cylinder inscrib'd in the same sphere. Fig. 26.

Let A K L be the square inscrib'd in the greatest circle of a sphere, from which turn'd round, there is describ'd a

ſquare cylinder; and let A L be drawn as a diameter common to the cylinder and ſphere. Becauſe the ſquare A L is (by 47. *l.* 1.) equal to the equal ſquares A K, K L, it will be double to one A K. Therefore alſo the circle of the diameter A L, is (by 2. *l.* 12.) double to the circle, whoſe diameter is A K; to wit, to the circle C N. But the ſuperficies of the ſphere is (by 24 of this) four-fold to the circle whoſe diameter is A L, for that is the greateſt circle of the ſphere, ſeeing A L is the diameter of the ſphere. Therefore the ſuperficies of the ſphere is eight-fold of the circle C N. But becauſe L K, K A (by the hypotheſis) are equal, the cylindrical ſuperficies A C L is (by *Coroll. p.* 12. of this) quadruple of the circle C N. Therefore ſince the ſuperficies of the ſphere is eight-fold of the ſame circle, it will be double to the cylindrical ſuperficies. *Q. E. D.*

PROP. XXXIV. Theorem.

Fig. 26. *THE ſuperficies of a ſphere hath that proportion to the whole ſuperficies of a ſquare cylinder inſcrib'd in it, which 4 hath to 3.*

Let the ſame things be ſuppos'd which were in the foregoing demonſtration. Becauſe, by the hypotheſis L K, the ſide of the cylinder, and A K the diameter of the baſe thereof, are equal, the cylindrical ſuperficies C L will be quadruple (by *Coroll. p.* 12. of this) to the baſe C N, and conſequently the whole ſuperficies of the cylinder is to both baſes C N and S L, as 6 is to 2. But the ſuperficies of the ſphere is to both baſes together, C N, S L, as 8 is to 2, ſeeing in the foregoing it was ſhew'd that it is to one baſe as 8 to 1. Therefore the ſuperficies of the ſphere is to the cylindrical ſuperficies C L, as 8 is to 6, or 4 to 3. *Q. E. D.*

Corollary.

THE whole ſuperficies of a right cylinder deſcrib'd about a ſphere, is to the whole ſuperficies of an equilateral cylinder inſcrib'd, as 2 is to 1. For the circumſcrib'd is to the ſpheric ſuperficies as 12 is to 8. by 32. of this.) But the ſpheric is to the inſcrib'd, as 8 is to 6 by this preſent propoſition. Therefore the circumſcrib'd is to the inſcrib'd, as 12 is to 6, or 2 to 1.

PROP.

PROP. XXXV. Theorem.

A Portion of any spherical superficies whatever (as ILBG) hath the same proportion to the superficies of the greatest inscribed cone, which (BG) the side of the cone hath to (GO) the radius of the base. Fig. 26, or 25.

Because (by 25. of this) the superficies of the portion ILBG is equal to the circle of the radius BG; the proportion thereof to QT, that is to the base of itself and of the cone, will be duplicate to the proportion of the conical superficies IBG to the same base QT. Therefore it is manifest, (by *Definition* 10. l. 5.) that the superficies ILBG is to the conical superficies IBG, as the same conical superficies IBG, is to the base QT. Wherefore seeing the conical superficies IBG, is to the base QT, as BG (by 14 of this) is to GO, the superficies of the portion will also be to the conical superficies IBG inscrib'd in it, as BG is to GO. *Q. E. D.*

PROP. XXXVI. Theorem.

THE superficies of the hemisphere (EOBD) hath that proportion to (EBD) the superficies of the greatest right inscribed cone, which in a square the diameter hath to a side; and that proportion to the superficies of a like cone circumscribed, as the side in a square hath to the diameter. Fig. 24.

I. The demonstration of the first part is manifest from the foregoing. For EOBD the superficies of any portion whatever, and consequently of the hemisphere, is to the conical superficies inscrib'd, as BD is to DA. But BADK is a square whose diameter is BD, and the side DA. Fig. 6. l. 4.

Part II. Let [illegible] be half of the square circumscrib'd about the circle, (whose center is O) which EBC being turn'd about the axis AB, let there from thence be produced a cone circumscribed about the hemisphere. Now, because the square EC is (by 47. l. 1.) double to the square EB or GI, the circle of the diameter EC also is (by 2. l. 12.)

l. 12.) double to the circle whoſe diameter is G I, that is, to the circle H G D I. But (by 24 of this) the ſuperficies of the hemiſphere included in the cone E B C, is double to the ſame circle. Therefore the circle of the diameter E C is equal to the hemiſpherical ſurface. Wherefore ſeeing the conical ſuperficies E B C is (by 14. of this) to the circle of the diameter E C, to wit, to its own baſe, as the ſide B E is to E O, the radius of the baſe; it will be alſo to the hemiſpherical ſuperficies inſcribed in it, as B E is to E O, that is, as the diameter in a ſquare is to a ſide. *Q. E. D.*

The ſame Figure with Fig. 13. l. 5.

PROP. XXXVII. Theorem.

A Sphere hath the ſame proportion to a ſquare conical rhombus circumſcribed about it, both in reſpect of the ſolidity and ſurface, which in a ſquare the ſide hath to the diameter.

Let the ſquare E B C F be circumſcrib'd about H G D I, the greateſt circle of a ſphere, from which ſquare, as turn'd round about the axis B F, let a conical rhombus encompaſſing the ſphere be produced.

As E B, a ſide of the ſphere (ſee *Fig.* 6. *l.* 4.) is to the diameteter E C, even ſo let S be made to R; ſee *Fig.* 13. *l.* 5.) and let this proportion be continued through four terms, S, R, Q, O, the proportion then of S to O will be triplicate to the proportion of S to R; that is, of E B to E C, and the proportion of O to R will be duplicate to the proportion of O to Q, or of R to S; that is of E C to E B; and conſequently (by 20. *l.* 6.) O is to R as the ſquare of E C is to E B; from whence (by *Schol. Prop.* 6. and 7. *l.* 4.) O is double to R. Theſe things being thus ſettled, let the ſphere E B C F be underſtood to be circumſcribed about the conical rhombus. Thus the ſphere H G D I will be to the ſphere E B C F (by 18. *l.* 12.) in the triplicate proportion of the diameter G I or E B to the diameter E C; that is, (as I have already ſhew'd) it will be as S to O. But the ſphere E B C F is to the conical rhombus inſcribed in it (by 30 of this) as 2 is to 1; that is, (as I have [illegible]) as O is to R. Therefore by equality of proportion, the ſphere H G D I is to the ſame rhombus which is deſcribed about it, as S is to R; that is, as in a ſquare the ſide E B is to the diameter E C. Which was the firſt part. Then from the ſecond part of the foregoing, it appears, that the ſuperficies

superficies of the hemisphere is to the superficies of the cone EBC, and consequently the superficies of the whole sphere is to the superficies of the whole rhombus EBCF, as in a square the side is to the diameter. Therefore the sphere, as well in solidity as in the superficies is to the square rhombus EBCF, as in a square the side is to the diameter. *Q. E. D.*

PROP. XXXVIII. Theorem.

THE superficies of the portion (BGKD) which contains an equilateral cone (BKD) is double to the superficies of the same cone. Fig. 27.

This is manifest from 35. For the superficies of the portion BGKD is to the inscrib'd conic superficies (by 35. of this) as BK is to BA. But because the cone BKD is suppos'd to be equilateral, KB is equal to BD, and consequently double to BA. Therefore the superficies BGKD is also double to the inscribed conical superficies. *Q. E. D.*

PROP. XXXIX. Theorem.

THE superficies of a sphere is to the whole superficies of an equilateral cone inscribed in it, as 16 *to* 9. Fig 27.

Let Z be the center of the sphere, and BKD the equilateral cone inscribed, and KZAO the axis common to the sphere and cone. If the sphere and cone be cut through this, there will be produced in the sphere the greatest circle, and in the cone, the equilateral triangle BKD, one side whereof will be the diameter of the basis of the cone QT. And because the axis of the cone KA is perpendicular to the base QT, BAK (*Def.* 3. *l.* 11.) will be a right angle. Therefore the square of BA is equal to the rectangle KAO. (*Coroll.* 1. *p.* 17. *l.* 6.) Now because the side of the equilateral triangle cuts off (*Coroll.* 5. *p.* 15. *l.* 4.) a fourth part of the axis AO, the rectangle KAO, that is, the square of BA, will be triple to the square of AO (by 1. *l.* 6.) Wherefore seeing the square of the radius ZO is (*Coroll.* 3. *p.* 1. *l.* 2.) quadruple of the square of AO, the square of the radius ZO will be to the square of the radius BA, as 4 is to 3.

Therefore the circle O B K D is alſo (by 2. *l.* 11.) to the circle Q T, as 4 is 3. Therefore four circles, O B K D, that is, (by 24 of this) the whole ſpherical ſuperficies D G is to the circle Q T, as 16 is to 3. But (*Coroll. p.* 14. of this) the ſuperficies of the equilateral cone B K D is to the circle Q T, to wit, its own baſe, as 2 is to 1; and conſequently the whole ſuperficies of the cone B K D; that is, including its baſe, is to the baſe, to wit, the circle Q T, as 3 is to 1, or 9 to 3. Therefore, ſeeing I have ſhew'd that a ſuperficies of a ſphere is to the ſame circle as 16 is to 3, the ſuperficies alſo of the ſphere D G will be to the whole ſuperficies of the equilateral cone, as 16 is to 9. *Q. E. D.*

Or otherwiſe thus.

BEcauſe, (by *Cor.* 5. *Prop.* 15. *l.* 4.) the ſide B D of the equilateral triangle cuts off a fourth part of the axis A O, the ſpherical ſuperficies B O D will be a fourth part by 27. of this, and conſequently the ſuperficies D G D K, three fourth parts of the ſuperficies of the whole ſphere. Wherefore if the whole ſuperficies be ſuppos'd to be 16, the ſuperficies B G K D will be 12. But (by the foregoing) the ſuperficies B G K D is double to the conical ſuperficies B K D, and conſequently is to it, as 12 to 6. Therefore the whole ſuperficies of the ſphere is to the conical B K D, as 16 is to 6. Then becauſe the ſuperficies of the cone B K D (as being equilateral) is (by *Cor.* 1. *Prop.* 14 of this) double to the baſe Q T, it is manifeſt, that the conical ſuperficies B K D (to wit, without the baſe) is to the whole ſuperficies of the cone, as 2 is to 3; that is, as 6 to 9. Therefore by equality of proportion, the whole ſuperficies of the ſphere is to the whole ſuperficies of the equilateral cone inſcrib'd, as 16 to 9. *Q. E. D.*

PROP. XL. Theorem.

Fig. 28. *THE ſuperficies of a ſphere bears the proportion to the whole ſuperficies of an equilateral cone circumſcribed about it, that 4 doth to 9.*

Let there be circumſcrib'd about the greateſt circle of a ſphere B P M, the equilateral triangle D O F, by which as turn'd round about the axis O A B, let there be produced an equilateral cone circumſcribed about the ſphere. And

let

let there also be circumscribed about the equilateral triangle D O F, the circle N D L O F, which, as is manifest, is concentrical to the former; and let the axis O A B be produced to N. Because B N is a fourth part of the axis O N, as is manifest from (Cor. 5. Pr. 15. l. 4.) O N is double to B K. Wherefore the proportion betwixt circles being duplicate (by 2. l. 12.) of the proportion of the diameters, the circle B P M will be to the circle N D L O F, as 1 to 4. But it hath already been shewed in the foregoing demonstration, that the circle N D L O F, is to the circle Q T, the base of the equilateral cone inscrib'd in the sphere F L, as 4 is to 3. Therefore, by equality of proportion, the circle B P M is to the circle Q T, as 1 is to 3. But the whole surface of the cone D O F is (by *Cor.* 1. *Prop.* 14. of this) triple to Q T. Therefore the whole superficies of the cone is nine-fold of the circle B P M. Wherefore seeing the superficies of the sphere T P is quadruple (by 24. of this) of the same circle B P M, the whole superficies of the equilateral cone D O F is to the superficies of the sphere to which it is circumscrib'd, as 9 to 4. *Q. E. D.*

Corollary 1. " From this demonstration it is manifest " that the axis B O, of an equilateral cone, circumscrib'd " about a sphere, is one and a half of the diameter of the " sphere E K, or as 3 to 2.

2. " That Q T, the base of the cone D O F, is also " one and an half of both bases of the cylinder circum- " scrib'd about the same sphere. For Q T is to B P M, " as 3 to 1. Therefore Q T is to B P M twice, as 3 is to 2.

3. " That the superficies of the cone D O F is one and " an half of the superficies of the equilateral cylinder cir- " cumscrib'd about the same sphere. For that * is double " to Q T, while this is quadruple to B P M §. Therefore " the conical superficies will be to the cylindrical, as twice " 3 to 4 times 1; that is, as 6 to 4, or as 3 to 2.

* *Per Cor.* 1. *p.* 31.
§ 24 *and* 26 *of this.*

PROP. XLI. Theorem.

THE whole superficies of an equilateral cone circumscribed about a sphere, is quadruple to the whole superficies of a cone inscribed in the same sphere. Fig. 28.

By the foregoing, the whole superficies of the equilateral cone DOF circumscrib'd, is to the superficies of the sphere,

as

as 9 to 4; and the ſuperficies of the ſphere is to the whole ſuperficies of the inſcribed cone S K T, as 16 to 9 (by 39. of this.) Therefore by permutation of equality of proportion, the whole ſuperficies of the circumſcribed equilateral cone is to the whole ſuperficies of the equilateral inſcribed, as 16 to 4, or as 4 to 1. *Q. E. D.*

PROP. XLII. Theorem.

Fig. 29.

A *Sphere hath that proportion to B K C, an equilateral cone inſcribed in it, which 32 hath to 9.*

Let the ſphere and cone be cut by a plane paſſing through the common axis K O, producing in the ſphere the greateſt circle O F K I, and in the cone the equilateral triangle B K C. Then a plane being drawn through the center A, perpendicular to O K, let the hemiſphere F G K I be cut off, in which let the greateſt cone F K I be underſtood to be inſcribed. Now becauſe (by *Cor.* 5. *p.* 15. *l.* 4.) the ſide B C of the equilateral triangle cuts off a fourth part of the axis O K, P K will be to A K, as 3 to 2, that is, as 9 to 6. But the baſe Q T is to the circle O F K I, that is, to the baſe N D, as 3 to 4, that is, as 6 to 8, as appears from what was demonſtrated, *Prop.* 39. Wherefore ſeeing the proportion of the cone B K C to the cone F K I, is (by *Schol.* 2. *p.* 15. *l.* 12.) compounded of the proportion of the altitude P K to the altitude A K (that is, of the proportion of 9 to 6,) and of the proportion of the baſe Q T to the baſe N D (that is, of the proportion of 6 to 8) the cone B K C will be to the cone F K I, as 9 to 8. Wherefore ſeeing (by 30 of this) the ſphere C G is quadruple of the cone F K I, the equilateral cone B K C will be to ſphere C G, as 9 to 32. *Q. E. D.*

PROP. XLIII. Theorem.

Fig. 28.

A*N equilateral cone circumſcribed about a ſphere, is eight-fold of an equilateral cone inſcribed in the ſame ſphere.*

Let S K T and D O F be the equilateral cones inſcrib'd and circumſcrib'd, and let O K B be the common axis, Then let as well both the cones as the ſphere be cut by a plane

plane paſſing through the axis; their ſections will be two equilateral triangles, and the greateſt circle B P M. About the triangle D O F likewiſe let there be underſtood to be deſcrib'd the circle N D O F, and let the axis O K B be produced unto N. Now becauſe the ſide D F of the equilateral triangle doth (by *Cor.* 5. *p.* 15. *l.* 4.) cut off N B, the fourth part of the axis O N, it is manifeſt that O N is double to B K. In like manner, becauſe the ſide S T of the other equilateral triangle cuts off B C, the fourth part of the axis B K, N O will be to B O, as F K is to C K; and by changing, as N O is to B K, ſo is B O to C K. But NO is double to B K. Therefore B O is likewiſe double to C K. Therefore becauſe of the ſimilitude of the triangles, D O F, S K T, D F and S T alſo, to wit, the diameters of the conical baſes will (by 4. *l.* 6.) be in a double proportion betwixt themſelves. Wherefore ſeeing the cones D O F, S K T, be like, and conſequently by (12 *l.* 12.) their proportion is triplicate to the proportion of the diameters D F and S T; which is that of 2 to 1, the cone D O F will be to the cone S K T, as 8 to 1. *Q. E. D.*

PROP. XLIV. Theorem.

A *Sphere hath the ſame proportion both in reſpect of ſolidity and ſurface to the equilateral cone D O F circumſcribed about it, which* 4 *hath to* 9. Fig. 28.

The ſphere T P is (by 42 of this) to the equilateral cone S K T inſcribed in it, as 3 is to 9. But (by the foregoing) S K T, the equilateral cone inſcribed, is to D O F, the equilateral cone circumſcribed, as 1 is to 8, that is, 9 to 72. Therefore by equality of proportion, the ſphere T P is to D O F, the equilateral cone circumſcribed, as 32 is to 72, that is, as 4 to 9. But in *Prop.* 40 we demonſtrated that the ſuperficies of a ſphere is to the whole ſuperficies of an equilateral cone circumſcribed, as 4 is to 9. Therefore a ſphere, both in ſolidity and ſuperficies, is to an equilateral cone circumſcribed about it, as 4 is to 9. *Q. E. D.*

That therefore which *Archimedes* was ſurpriz'd at in a ſphere and cylinder encompaſſing it, we have alſo now demonſtrated in a ſphere and equilateral cone encompaſſing it, to wit, that there is the ſame rational proportion of the ſolidities betwixt themſelves which there is of the ſurfaces.

For

For as he found that the ſphere is to the cylinder, as well in ſolidity as ſuperficies, as 2 to 3; ſo we have now taught, that the ſphere is, in reſpect both of ſolidity and ſurface, to an equilateral cone encompaſſing, as 4 to 9.

But from hence we ſhall, without much labour, demonſtrate, that the very proportion, to wit, the ſeſquialteral, which *Archimedes* ſhew'd to be betwixt the ſphere and cylinder, is continued by the equilateral cone circumſcribed both in the ſolidity and ſuperficies; and ſo we ſhall put an end to the preſent work.

PROP. XLV. Theorem.

See the Figure prefixed to this Treatiſe.

AN equilateral cone circumſcribed about a ſphere, and a right cylinder in like manner circumſcribed about the ſame ſphere, and the ſame ſphere, itſelf, continue the ſame proportion; to wit, the ſeſquialteral, as well in reſpect of the ſolidity as of the whole ſuperficies.

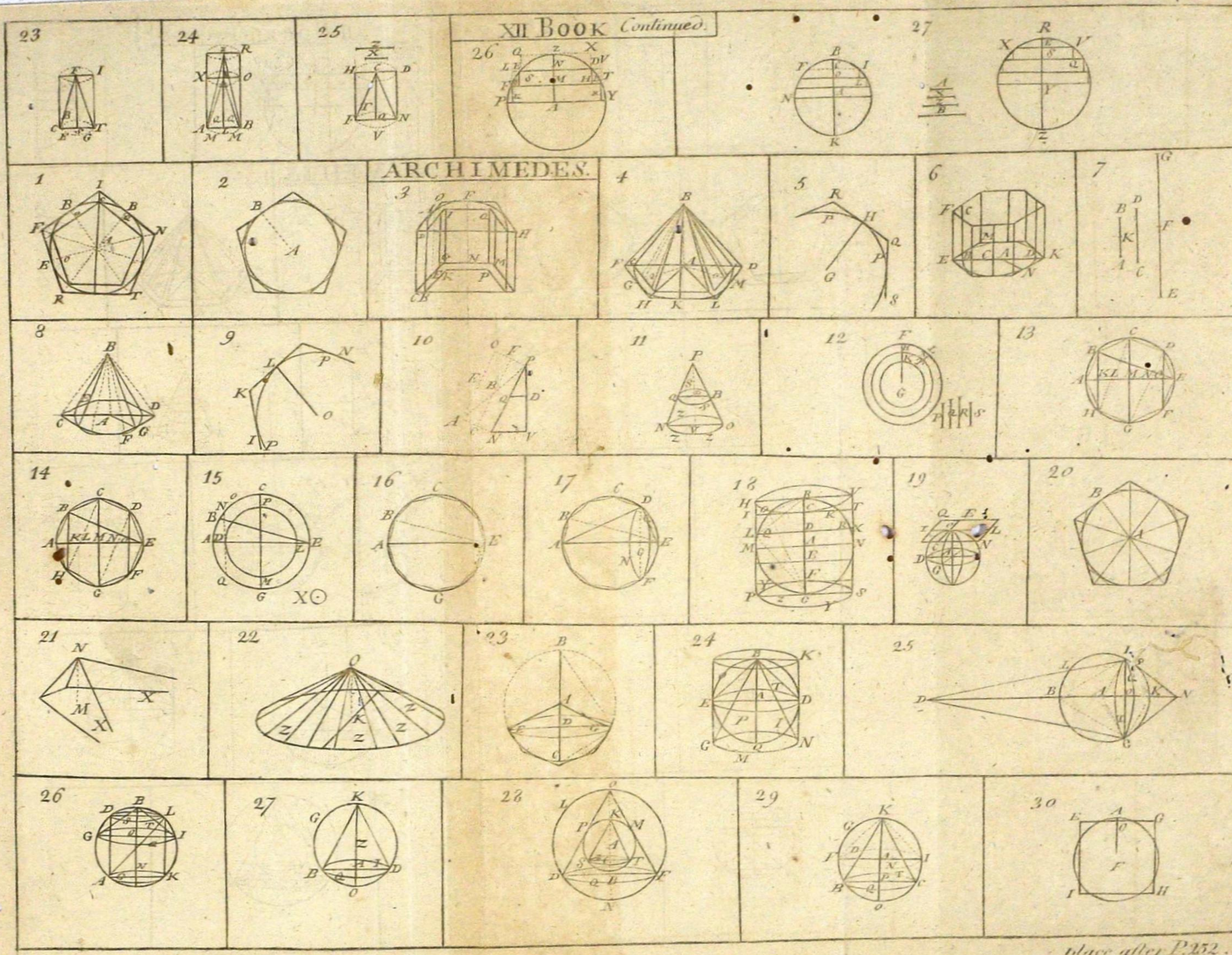
For by 32. of this book, the right cylinder G K encompaſſing the ſphere, is to the ſphere, as well in reſpect of ſolidity, as of the whole ſuperficies, as 3 is to 2, or as 6 to 4. But by the foregoing, the equilateral cone BAD circumſcribed about the ſphere, is to the ſphere in both the ſaid reſpects, as 9 is to 4. Therefore the ſame cone is to the cylinder both in reſpect of ſolidity and ſurface, as 9 is to 6. Wherefore theſe three bodies, a cone, cylinder and ſphere, are betwixt themſelves, as the number 9, 6, 4, and conſequently continue the ſeſquialteral proportion. *Q. E. D.*

[*PROP. XLVI.*]

THE ſame ſeſquialteral proportion holds betwixt an equilateral cone and cylinder circumſcribed about the ſame ſphere, in reſpect of their whole ſurfaces, their ſimple ſurfaces, their ſolidities, altitudes and baſes.

This propoſition is manifeſt as to the whole ſurfaces and ſolidities from the foregoing; as to the ſimple ſurfaces, from Cor. 3. Pr. 40. *of this; as to their altitudes and baſes, from* Cor. 1. *and* 2. *of the ſame 40th propoſition.*

FINIS.



place after P.252.

AN

APPENDIX

OF

PRACTICAL GEOMETRY.

In TWO PARTS.

The FIRST contains an *Easy*, *Brief* and *Independent* Demonstration of certain Select and most Useful *Propositions* in EUCLID.

The SECOND contains a *Synopsis* of all the *Problems* in the preceding Book; with a *Supplement* of sundry choice *Problems* in Practical Geometry.

For the Use of young Students in the Mathematicks.

By S. F.

DUBLIN:

Printed in the Year M,DCC,XCI.

APPENDIX.

PART I.

WE often find that EUCLID, for the demonstrating one *important* proposition, hath made use of a long chain of others, which have no other end, but to demonstrate that *principal* one: If we can *all at once* demonstrate those *capital* propositions, without such a *series* of preparatory demonstrations; we shall doubtless retrench many useless things, gain time, and render this appendix of some service to the young *student*.

There are primary propositions, of which *Des Cartes* writes in a letter not yet printed, as hinted in *Page* 65. of Dr. PELL's *Algebra*; *In searching the solution of Geometrical Questions, I always make use of lines, parallel and perpendicular, as much as possible,* (i. e. *as many lines as are useful) and I consider no other* Theorems, *but the* 47th, L. 1. *and* 4th, L. 6, *of* Euclid, *and I am not afraid to suppose many unknown quantities, that I may reduce the proposed question to such terms, as to depend on no other* Theorems, *but these two.*

Observe

Obſerve in how great eſteem the ingenious *Des Cartes* held theſe two *Theorems*; to the firſt of which, moſt of the preceding propoſitions in EUCLID are preparative, and which we ſhall eſſay to demonſtrate *independently* of any of them.

E U C L I D.

B O O K I.

PROP. XLVII. Theorem.

IN every right-angled triangle (A B C) the ſquare of the ſide (A C) which is oppoſite to the right-angle, is equal to the two ſquares of the other ſides (A B and B C) taken together.

In this are two varieties; firſt, when the two ſides A B and B C are equal. *Fig. 1.*

On the ſide A C erect the ſquare A C D E, on the ſide A B erect the ſquare A B D G, on the ſide B C erect the ſquare B C E F.

Demonſtration. 'Tis evident at firſt view of *Figure* 1. that one half of the ſquare of A B or B C, is an exact quarter of the ſquare of A C, conſequently four halves of the ſquares of A B and B C, which are both their whole ſquares, are equal to four quarters of the ſquare of A C, which is the whole ſquare of A C. *Q. E. D.*

Secondly, when the ſides A B and B C are unequal. *Fig. 2.* Make two equal ſquares B D E F. B D G H on the line B D, equal to A B and B C, the two leſs ſides of the given triangle; in the firſt, on A C, erect the ſquare A C I K; in the other, on A B and B C, erect the ſquares A D B L =□B C and B C H M=□A B; and laſt of all, draw the lines A C and B G.

Demonſtration. Firſt, 'Tis evident from the conſtruction that the ſquares B D E F and B D G H are equal.

Secondly, 'Tis obvious at firſt view that the ſquare B D E F exceeds the ſquare of A C by four right-angled triangles, whoſe baſe is $=$A B and perpendicular $=$B C, and conſequently, by *Axiom* 7. equal to one another, and the given triangle A B C.

Thirdly, 'Tis equally evident, B D G H exceeds the ſquares of A B and B C, by four right-angled triangles, whoſe baſe is $=$A B and perpendicular$=$ B C, and conſequently, by *Axiom* 7. equal to one another, and the given triangle A B C, and alſo to the four right-angled triangles aforeſaid.

Fourthly, Wherefore, by *Axiom* 3. if from equals, (the ſquares B D E F and B D G H) be taken away equals (the four right-angled triangles in each ſquare) the remains will be equal (the ſquare of A C in one equal to the ſquares of A B and B C in the other.) *Q. E. D.*

Or Algebraically, thus :

Let the biggeſt ſquare made of the ſum of the ſides $=SS$

$$\left.\begin{array}{l}\square AC=HH\\ \square AB=BB\\ \square BC=PP\end{array}\right\}\text{ I ſay, } HH=BB+PP.$$

1. $SS=HH+2BC$, which $2BC=4$ triangles,
2. $SS=BB+PP+2BC$, by $B+C\times B+C$.
3. $HH+2BC=BB+PP+2BC$.
4. $HH=BB+PP$. *Q. E. D.*

A like uſeful theorem of ſignal ſervice in the whole theory of compound motions, I ſhall ſubjoin.

In every parallelogram the ſum of the ſquares of the two diagonals is equal to the ſum of the ſquares of the four ſides.

To prove this by trigonometry requires 21 operations, by analyſis, or algebra 15. which *M. de Lagny* has reduced to 7 ſteps.

Howbeit

Howbeit the reaſonableneſs of this theorem may appear *Fig.* 3. from this ſingle conſideration, that all triangles on the ſame or equal baſe, and betwixt the ſame parallels, are equal, compared with the 12th and 13th of the ſecond book.

In this are two varieties; firſt, when the parallelogram is right angled.

Then the propoſition is evident from the 47th, I. *Euclid* juſt now demonſtrated.

Secondly, But when the angles of the parallelograms are oblique, draw the prick'd lines A G, B F, C H, D E, which being perpendiculars betwixt the ſame parallels, are all equal. Alſo G B and D H, which are alſo equal to A F and E C, being perpendicular betwixt the equal parallels, let each of the firſt 4 $=y$ and each of the laſt 4 $=z$, alſo A D $=$ B C call x; conſequently B E $=x-z$.

Demonſtration.

4 Sides.	□ B C $=xx$.	2 Diag.	□ B D $=yy+xx+zz-2xz$.
	□ A D $=xx$.		□ A C $=yy+xx+zz+2xz$.
	□ A B $=yy+zz$.		
	□ D C $=yy+zz$.		

Sum $2yy+2xx+2zz$. $=$ Sum. Dia. q. $2yy+2xx+2zz$.

Corollary. " Hence 'tis plane, the ſquare of the longer " diagonal exceeds the ſum of the ſquares of the two " contiguous ſides, exactly by as much as the ſquare of " the ſhorter diagonal wants thereof; that is to ſay, by " the double rectangle xz, whoſe length is the longeſt " ſide, and breadth equal to the diſtance that the perpen- " dicular from the oppoſite angle falls from one end of it, " either within or without the parallelogram.

EUCLID. Book VI.

PROP. IV. Theorem.

TRiangles which are equiangular to one another, are like or similar, that is, have their sides also that are opposite to the equal angles proportional.

Fig. 4. This I take to be the same with the first definition of the same book.

And that similar triangles (ABC, ADE) have their corresponding sides proportional, I shall illustrate from *Figure* 4.

If the triangle ABC be laid on the triangle ADE, so will the angle A, because equal in both, exactly coincide, and the line AC fall on the line AE, and AB on AD, by *Axiom* 7. and because the angle C=angle E, and the angle B=angle D, the side BC will be parallel to DE, by *Defin.* 1. *Book* 6.

Then suppose AB a third of AD, make BF another third, and parallel to DE draw FG, then parallel to AD draw GH and CI; by which parallels, the sides AE and DE will also be divided into three equal parts, by *Axiom* 12. Therefore it will hold as,

AB : AD :: AC : AE 4 12 5 15 that is, AB : 3AB :: AC : 3AC	and	AC : AE :: AB : AD 5 15 4 12 that is, AC : 3AC :: AB : 3AB
AB : AD :: BC=DI : DE 4 12 3 9 that is, AB : 3AB :: DI : 3DI.		CB=DI : DE :: AC : AE 3 9 5 15 that is, DI : 3DI :: AC : 3AC.

Q. E. D.

To

To trisect an arch of a circle B C.

This has been by sundry antient and modern geometricians accounted impossible to be done by the *Euclidean* geometry, which makes use of only a circle and straight lines; howbeit, we will attempt it, and afterwards a demonstration thereof.

Quarter the circle, and extend the diameter B G to P; parallel to it, draw C X and C R parallel to E X; then take the diameter B G in the compasses and move the ruler on the point C, till F P be exactly equal to B G, then draw C F O P, so will G O be a third of B C. *Fig.* 5.

Demonstration.

Through the center draw O E Z. Now because E O is radius and F P the diameter, and the angle F E P is right; therefore the lines F O, E O, P O, are all equal, and also the external angle F O E=2 P E O the two internal opposite angles, or 2 Z E B, which is vertical, and consequently equal to P E O. But F O E, that is, C O Z: being in the periphery, is measured by half the arch C Z; wherefore B Z, which is the measure of half the angle C O Z, is a fourth of the whole arch C Z. And consequently a third of B C; and therefore G O=B Z is also a third of B C. *Q. E. D.*

Note, If the arch to be trisected be greater than a quadrant, than trisect its complement to 180, and the third of this complement, taken from 60 degrees always leaves the third of the arch required.

Lest the foregoing demonstration should appear too concise to some, I will attempt it after another manner, from the following *Lemma*.

"That the measure of the angle C P B, at a point "without the circle is equal to half the difference betwixt "the intercepted arches C B and O G, or to the difference betwixt half their sum C O Z, and the less arch "O G, which is the same.

For

For BZ=OG, because vertical angles, by 15 *l.* 1. then is CZ the sum of the intercepted arches CB and OG, and COZ, the angle at the circumference, the measure of half the said sum, by 20. *l.* 3. which COZ, being external, is equal to the two internal and opposite angles OEP and OPE, by 32. *l.* 1. but the measure of OEP is OG, the less arch: Now, if COZ=OG, and the angle OPE=CPB, I say, CPB must be equal to the difference betwixt COZ and OG. *Q. E. D.*

Corollary. "In the said triangle EOP, If the angles "at E and P be both equal, then will OG be a third of "BC; because OG will be half of COZ, or a fourth of "the whole arch CZ, and consequently a third of BC "$=\frac{3}{4}$CZ, for from the whole CZ, take away BZ$\frac{1}{4}$, "remains BC$=\frac{3}{4}$. It remains only to prove the angles E "and P are equal.

Because FEP is a right-angle by construction the center of the semi-circle FEP will fall in O, which bisects FP, the diameter, which is double EO, the radius; consequently all three, FO, EO, PO, are equal radius's of the semi-circle FEP aforesaid; and because EO and PO are equal, the angles at the base E and P, by 5. *l.* 1. are equal.

A Synopsis *of the most useful and famous* Propositions *in* EUCLID's *first six books.*

BOOK I. has *Propositions* 32, 35, 37, 41, 44, 45, 47.
II. ——— 4, 5, 6, 12, 13, 14.
III. ——— 16, 20, 21, 22, 31, 32, 35, 36.
IV. ——— 10, 11, 12, 13, 14, 15, 16.
V. ——— {15, 16, 17, 18, or the foregoing compend. thereof.
VI. All the first 6, 8, 13, 14, 15, 19, 31.

Extra-

Extraordinary Propositions *not to be met with in* EUCLID, *whose demonstrations are omitted on purpose to exercise the genius of young mathematicians.*

1. IN every triangle the rectangle of any two sides is equal to the rectangle of the perpendicular from the said angle, and the diameter of the circumscribing circle, *i. e.* as perpendicular : one side : : other side : diameter.

2. The area of any triangle about a circle is equal to the rectangle of the semi-diameter, and half the sum of the sides.

3. An hexagon inscribed is a mean betwixt a trigon inscribed, and a trigon circumscribed, *Et sic de paribus.*

4. In any triangle the difference of the squares of two sides is equal to a rectangle of the base, and that segment of the base, which parted, in the middle of the other part the perpendicular falls.

5. The square of the mean, and the square of half the difference of the extremes together, is equal to the square of the half sum of the extremes.

6. If in a circle two lines be inscribed, intersecting each other, the rectangles of the segments of each line are equal; and the angle at the point of intersection is measured by half the sum of its intercepted arches.

7. If to a circle two right lines be adscribed, from a point without the rectangles of each line, from the said point to the convex and concave are equal, and the angle at the point is measured by half the difference of the intercepted arches.

8. If in a circle three right lines shall be inscribed, one of them cutting the other two, then the rectangles of the segments of each line so cut, are directly proportional to the rectangles of the respective segments of the said cutting line.

9. If a plain triangle be inscribed in a circle, the angles are one half of what their opposite sides do subtend; and if it hath one right angle, the longest side of that triangle shall be the diameter of the circle.

Eaſy Equations *ariſing by comparing two unequal* ſitions *of the Second Book of* EUCLID *are invented.*

The Reaſon.	Numb Equat.	In Species.	In Numbers.
	1	$z=m+n$	$10=7+3$
	2	$x=m-n$	$4=7-3$
$1-n$	3	$m=z-n$	$7=10-$[illegible]
$1-m$	4	$n=z-m$	$3=10-7$
$2+n$	5	$m=x+n$	$7=$ [illegible]$+3$
$2-x$	6	$n=m-x$	$3=7-4$
$1+2$	[illegible]	$2m=z+x$	$14=10+4$
$1-2$	[illegible]	$2n=z-x$	$6=10-4$
$7 \div 2$	9	$m=\frac{z+x}{2}$	$7=\frac{10+4}{2}$
$8 \div 2$	10	$n=\frac{z-x}{2}$	$3=\frac{10-4}{2}$
$7-x$	11	$z=2m-x$	$10=14-4$
$7-z$	12	$x=2m-z$	$4=14-10$
$8+x$	13	$z=2n+x$	$10=6+4$
$3-2n$	14	$x=z-2n$	$4=10-6$
$2\times x$	15	$xx=xm-xn$	$16=28-12$
3×4	16	$mn=zz+mn-zm-zn$	$21=100+21-70-30$
5×5	17	$mm=xx+2xn+nn$	$49=16+24+9$

Quantities and Numbers; whereby most of the Propo-
Let m=7, n=3, z=10, *and* x=4.

The Reason.	Numb. Equat.	In Species.	Propositions of the Second Book.
1×b	18	$zb=bm+bn$	1
1×z	19	$zz=zm+zn$	2
1×m	20	$zm=mm+mn$	3
1×n	21	$zn=nn+mn$	
1×1	22	$zz=mm+2mn+nn$	4
2×2	23	$xx=mm-2mn+nn$	
7×8	24	$4mn=zz-xx$	
24 $\frac{\because}{4}$	25	$mn=\frac{zz}{4}-\frac{xx}{4}$	
25 + $\frac{xx}{4}$	26	$\frac{zz}{4}=mn+\frac{xx}{4}$	5
1× 2	27	$zx=mm-nn$	
27+nn	28	$zx+nn=mm$	6
3×3	29	$mm=zz-2zn+nn$	
29+2zn	30	$mm+2zn=zz+nn$	7
24+xx	31	$4mn+xx=zz$	8
22+23	32	$zz+xx=2mm+2nn$	10
32 $\frac{\because}{2}$	33	$\frac{zz+xx}{2}=mm+nn$	9
		And so infinitely.	

PRACTICAL GEOMETRY.

PART II.

AS to the demonſtration of theſe *Practical Problems*, I have purpoſely omitted them, that the young ſtudent may exerciſe his own genius, in making application of, and recollecting what he has already learned from the foregoing books of *EUCLID*; to awaken the mind, to whet the appetite of our mathematical ſtudent, and to amuſe and improve him in eaſy and practical problems of geometry, is the main deſign of this appendix.

And leſt we go over what is already done, I think it not amiſs to give a *Synopſis* of all the *Practical Problems* both in the *Propoſitions* and *Corollaries* of the foregoing book, referring to the book and propoſition where they may be found.

In the First Book.

PROPOSITION 1. On a given line to make an equilateral triangle.

To meaſure inacceſſible diſtances, ſee 4. and 26. alſo *L.* 6. *Prop.* 8.

2. To draw a line equal to a given one.
3. From a greater to cut off a leſs.
4. To play at billiards.
6. To meaſure acceſſible altitudes, ſee 33. alſo *L.* 6. 4.
9. To biſect an angle.
10. To biſect a line.
11. To erect a perpendicular, and L. 3. 31.
12. To let fall a perpendicular.
15. That rays of light reflected, take the ſhorteſt courſe.
19. A globe can reſt no where, but in the point it touches the earth.
22. To make a triangle of three lines given.

23. To make an angle equal to a given one.
To measure a given angle.
To lay down an angle of any number of degrees.
27. To measure the compass of the earth, see *L.* 2. 6.
31. To draw parallels.
32. To determine the parallax of the stars.
To find the number of right angles contained in the angles of any right-lined figure.
33. The demonstration of compound motions.
34. To divide the area of a parallelogram.
36. Figures of equal compass may have different areas.
38. To divide the area of a triangle; that bodies move equal areas in equal times.
40. And are moved by a centripetal force.
41. To find the area of a triangle *L.* 2. 13.
42. To make a parallelogram with an angle equal to a given one, and equal to a triangle given.
44. On a given line, to make a parallelogram equal to a triangle given, and to have an angle equal to one given.
To demonstrate geometrical division.
45. On a given line, and with a given angle, to make a parallelogram equal to any straight-lined figure.
To find how much one straight-lin'd figure exceeds another.
46. To make a square on a given line.
47. To add any number of squares.
To take a less square out of a greater; any two sides of a right-angled triangle, to find the third.
The origin of the table of sines tangents and secants. See *L.* 3. 3. *L.* 4, &c.

In the SECOND BOOK.

11. To cut a line in extreme and mean proportion, *L.* 6. 30.
14. To find a square equal to any right-lined figure.

In the THIRD BOOK.

1. To find the centre of a circle.
12. To find the point where two circles touch each other.
16. To demonstrate the infinite divisibility of a straight line. See *L.* 1. 47.
17. To draw a tangent to a circle.

20. To demonſtrate, the ſides of triangles are in ſuch proportion as the ſines of their oppoſite angles.
To meaſure the diſtance of the ſun or moon.
21. How the ſame line, at different diſtances, may appear of the ſame length.
22. About what quadrangle a circle can or cannot be deſcribed, L. 6. 4.
25. To perfect an arch into a circle.
30. To biſect a given arch.
31. To prove whether a ſquare be true.
33. On a line to form a ſegment capable of any given angle.
34. From a circle, to cut a ſegment containing any given angle.
35. Certain geographical paradoxes ſolved; to draw a circle through any two points in another given circle, which ſhall diametrically cut it.

In the FOURTH BOOK.

1. To inſcribe a line in a circle.
2. To inſcribe a triangle in a circle.
3. To circumſcribe a triangle about a circle.
4. To inſcribe a circle in a triangle.
5. To circumſcribe a circle about a triangle.
6, 7. To inſcribe and circumſcribe a ſquare in or about a circle.
8, 9. To inſcribe and circumſcribe a circle about a ſquare.
10. To make an iſoſceles-triangle, whoſe angle at the baſe is double the angle at the vertex.
11. To inſcribe a regular pentagon, or any other polygon in a circle: On a given line to deſcribe a pentagon.
12. To circumſcribe a regular pentagon, or any polygon.
13, 14. To inſcribe in, or circumſcribe a circle about a regular polygon.
15. On a line given to deſcribe an hexagon, or to inſcribe it, or an equilateral triangle in a circle.
16. To inſcribe a regular quindecagon, or innumerable regular figures.
On a given line to deſcribe a regular figure,
What number of regular figures will fill a ſpace, *i. e.* whoſe angles about one point, juſt make 360 degrees.

In the SIXTH BOOK.

1. To divide a trapezium.
6. To make similar triangles.
9. To divide a line in a given proportion.
10. To divide a line as another is, or into any number of parts equally.
11. To find a third proportional to two given lines, also the sum of infinite proportionals.
12. To find a fourth proportional to three lines given.
13. To find a mean proportional, or two means betwixt two given lines divers ways, *L.* 12. 18.
14. To demonstrate the inverse rule of proportion.
16. To demonstrate the direct rule of proportion.
18. On a right line to describe a polygon like and alike situate, to a given one. Hence the drawing and reducing maps. See 20, 21.
19. Similar figures are in a duplicate ratio of like sides. See 20. 23. *L* 12. ...
25. To make a polygon equal to a given one, and like to another given one.
31. To add or subtract right-lined figures, and to square the lunets of *Hippocrates*.

In the ELEVENTH BOOK.

11. To draw a perpendicular to a plane.
12. To erect a perpendicular to a plane.
21. A demonstration of the five regular bodies.
33. Like bodies are to each other, as the cubes of their homologous sides, *L.* 12. 9.

In the TWELFTH BOOK.

18. To encrease or diminish solids.

By this synopsis, and the following appendix, it will appear what a large *Apparatus* towards a system of *Practical Geometry* is already extant in our language for want of which there has been these many years past, no small complaint: Especially if to this be added *Le Clerc*'s geometry, *Hawney*'s complete measurer, and *Langley*'s Practical geometry; to the first of which, I am not a little in debt for several of the following problems.

Problem

Problem I.

TO *divide a right line (A B) into any number of equal parts, ſuppoſe* 8.

For this purpoſe have in readineſs a number of equidiſtant parallels drawn and numbered from 0 to 12 or 20, with the diſtance of A B given. Set one foot of the compaſſes on the parallel marked 0.0. and turn the other till it touch the eighth parallel, unto which two feet apply a ruler, and draw the line A B, which the parallels will equally divide into eight parts. *W. W. D.*

This is ſo plain and eaſy, as not to need any figure.

Problem II.

TO *find the area of a plain triangle, having the three ſides, without a perpendicular.*

Add the three ſides, and take half that ſum; then ſubtract each ſide ſeverally from that half ſum: Multiply the half ſum and the three differences continually, and out of the laſt product extract the ſquare root, which is the area of the given triangle.

Or,

Having the diameter of the inſcribed circle, multiply it by a fourth of the ſum of the three given ſides for the area ſought.

Problem III.

HAVING *the area of any triangle, to find the diameter of the inſcribed circle.*

Divide the ſaid area by a fourth of the ſum of the three ſides; the quotient is the diameter of the inſcribed circle.

Problem IV.

TO *reduce any figure into a ſquare; or to make a ſquare equal to a parallelogram, triangle, trapezium, polygon or circle, &c.*

By the rules of menſuration, caſt up their areas, the ſquare root whereof is the ſide of the ſquare ſought.

Or

Or Geometrically thus:

Consider what two things multiplied together produce their area: Between those two numbers, or sides, find a geometrical mean by *Lib.* 6. 13 *Euclid.* This mean is the side of a square equal. *Q. W. D.*

N. B. The two things which multiplied, reduce their area in a	Parallelogram rectangled, Rhombus, Rhomboides,	are	The length and perpendicular breadth.
	Triangle,		Base and half perpendicular.
	Trapezium,		Diagonal and half of both perpendiculars.
	Circle,		Half diameter and half periphery.
	Polygon,		Half perimeter and the perpendicular from the center.
	Sector,		Half the arch and half the diameter.
	Segment,		Chord, and two thirds of the V. sine.

N. B. If it be an irregular figure, reduce it into triangles, or trapeziums, by drawing lines within from corner to corner; then reduce the said triangles, &c. into squares, which squares add by 47. 1. or 14. 2. *Euclid.*

Problem V.

T O *find a straight line* (E I) *nearly equal to the arch of a quadrant* (F C).

Fig. 6. Bisect the quadrant B D in F, and draw B F; to B F erect the perpendicular A F on B; then set A C from C to I, so will E I be nearly equal to the arch of the quadrant B D or B C.

Or,

Bisect B D in F, and B ... then with the distance K F, on the center K, draw the arch H F G I, so will the line H I be nearly equal to the semi-circumference, and E I to the quadrant, as before.

N. B. *Archimedes* demonstrated, that the circumference is to the diameter less than as 22 : 7. and greater than $21\frac{70}{71}$ to 7; within which stated limits, many years ago, it was found in whole numbers to be as 9 : 10 : : chord : arch of a quadrant.

Suppose then that the diameter = 7, its half 3.5, the square of which is 12.25; which doubled, is 24.5, whose square root is 4,949, the chord of the quadrant; then say, as 9 : 10 : : 4×4.949 : 21.995, the circumference, which is greater than $21\frac{70}{71}$, or 21.985. Again, 4.95 is something greater than the chord, which × 40, and divide by 9, as above, gives 22; so that by this proportion, as 9 : 10. a like answer with *Archimedes* is also discovered, *i. e.* less than as 22 : 7, and greater than as $21\frac{70}{71}$ to 7.

Con-

Concerning squaring the circle, I offer some few easy and practical observations following.

1. A Circle is equal to a right-angled triangle, whose base is the circumference of the circle, and its perpendicular the radius of it.

2. Every polygon circumscribed is greater, and every polygon inscribed is less than the circle.

3. The compass of a polygon circumscribed is greater, and inscribed is less than the circumference of a circle.

4. This right-angled triangle aforesaid, will be less than any polygon circumscribed, and greater than any inscribed: Because the circumference of this circle (which is the base of this triangle) is greater than the compass of any inscribed, and less than the compass of any circumscribed polygon; therefore it will be equal to the circle.

Because every imaginable inscribed figure, which is less than the circle, is also less than the triangle; and every circumscribed figure, greater than the circle, is also greater than the said triangle likewise, therefore the said triangle is equal to the circle: But actually to find a right line exactly equal to the circumference, is not yet discovered geometrically.

Howbeit, I shall offer an easy approximation to the young student, whom I desire to account the circle a polygon of 1000.10000 or 43200 sides, *i. e.* the half minutes in 360, so small a part of a circle will insensibly approach to, or become very nearly a strait line; nay, the product of the sine and tangent of one minute multiplied by the said 43200, will agree in the last 7 figures, *i. e.* 6283185, whereby we obtain the circumference of the circle, whose radius is 1000000, to the like number of figures exactly, and may to as many more places, if we add the two products together, and take the half for the circle's circumference, which will be less than the *Tangent Product*, or circumscribed polygon, and greater than the *Sine-Product*, or inscribed polygon.

For, according to the preceding third proposition out of *Archimedes*, the circuits or polygons circumscribed about and inscribed in a circle, do at last end in the circumference

cumference of the circle; in like manner the polygons themselves do at last end in a circle.

To this I subjoin certain practical remarks on regular polygons, their angles and sides.

REgular polygons may be delineated several ways; first, by the angle at the center; secondly, by the angle at the figure; thirdly, by the angle of the triangle at the base.

To find each

Divide 360 by the number of sides, the quotient is the angle at the center, whose complement to 180 is the angle at the figure, and half the said complement is the angle at the triangle. By this one may make a table of angles for as many polygons as he pleases.

Polygons of odd number of sides are inscribed in circles by the help of *Isosceles* triangles, whose angles at the base are multiples of that at the top.

If the angle at the base be	Double,	the angle at the *Vertex*, the base will be the side of a	Pentagon.
	Triple,		Heptagon.
	Quadruple,		Enneagon.
	Quintuple,		Endecagon.

The number of degrees of the angle at the *Vertex*, is found by dividing 180 by the number of *Sides* in the *Polygon* to be inscribed, the quotient gives the said angle; which doubled, tripled, &c. will give the angle at the base.

Seeing polygons inscribed are the chords of the angles at the center, which chords are always double the sine of half the angle at the center: Therefore in the table of natural sines, hunt out the sine of half the angle at the center, which doubled, is the exact side of the polygon, in such parts as the radius contains 10000000, &c.

A Table

A Tab[illegible] for the inſcribing [illegible]nd deſcribing Poly-gons.

Numb. of ſides.	Quantity of the ſide.	Angles at the [illegible]	Angles at the Figure.	Angles [illegible] the Triangle.
3	1732[illegible]508	1[illegible]	6[illegible]	30
4	14142[illegible]35	90	[illegible]	45
5	1175570[illegible]	72	108	54
6	10000000	[illegible]	120	60
7	86776[illegible]4	$51\frac{3}{7}$	$128\frac{4}{7}$	$64\frac{2}{7}$
8	765[illegible]68	45	135	$67\frac{1}{2}$
9	6840[illegible]02	40	140	70
10	6180[illegible]9	36	144	72
11	5634[illegible]	$32\frac{3}{11}$	$147\frac{2}{11}$	$73\frac{7}{11}$
12	51763[illegible]	30	150	75

The Uſe.

To inſcribe a heptagon in a'circle, whoſe radius is 500; ſay as *Tabular Radius* 1000 : *Tabular Heptagon* 867 : : ſo is the given radius : ſide heptagon ſought 433.

To deſcribe a heptagon on a given line, find the radius proper; ſay as the *Tabular Heptagon* 867 : *Tabular Radius* 1000 : : ſo is the given ſide 433 : to the proper radius ſought 500.

Problem VI.

TO make a circle (A D G) nearly equal to a given ſquare (A E C F.)

Draw the diagonal A C, which divide into 10 equal parts; 8 of them are nearly equal to the diameter of the circle ſought: Wherefore in the middle of the diagonal A C, and with the radius of 4 of thoſe equal parts draw the circle A D G. *W. W. D.* *Fig.* 8.

Or thus more briefly :

Biſect E C in D, and draw A D, the diameter of the circle ſought.

Problem VII.

TO make an Iſoſceles triangle (B C E) equal to a given ſquare (A B C D.)

S 2 Extend

Fig. 8. Extend the ſide C D to E, making D E=C D, and draw B E, ſo will B C E be equal to A B C D. *W. W. D.*

To make an equilateral triangle equal to a ſquare, ſome adviſe to make the ſide of the triangle one and a half of the ſide of the ſquare; but it is ſomewhat too little.

Problem VIII.

WIthin a given triangle (*KLM*) *to inſcribe a ſquare* (*NOPQ.*)

Fig. 9. On M erect the perpendicular M H, equal to L M; let fall the perpendicular K G, then draw G H, cutting K M in N; through N draw N O, parallel to the baſe, L M, and N Q, and O P parallel to K G, ſo is N O P Q the greateſt inſcribed ſquare. *W. W. D.*

Problem IX.

WIthin the ſquare (*RSTV*) *to inſcribe an equilateral triangle* (*RZY.*)

Fig. 10. Draw the two diagonals S T and R V; on V, as a center, with the extent V X, draw the arch Z X Y, then draw the lines R Z, Z Y, R Y, which form the triangle R Z Y. *W. W. D.*

Problem X.

ABout an equilateral triangle (*ABC*) *to deſcribe a ſquare* (*AFDG.*)

Fig. 11. Biſect B C in H, and draw A H D; make H D equal to H C, and biſect A D in E; through E draw F E G at right angles, making F E and E G each=E D, then draw the lines A F, F D, D G and A G, forming the ſquare. *W. W. D.*

Problem XI.

ABout a ſquare (*LMNO*) *to draw a triangle* (*PQR*) *whoſe angles ſhall be equal to the ſeveral angles of a triangle given* (*STV.*)

Extend

Extend N O to P Q, then make the angle R L M=T Fig. 12. and R M L=V, then draw the sides R L P and R M Q. So is R P Q the triangle. *W. F. D.*

Problem XII.

TO inscribe an equilateral triangle (*A F G*) in a given pentagon (*A B C D E.*)

Find its center H, and d[illegible] cle about it; then Fig. 13. inscribe a triangle in that cir[illegible] will also be the triangle in the pentagon. *W. W. D.*

Or,

With the radius A H, and center A, draw the arch H M, which bisect in K, and draw A K F, the side of the triangle sought.

Problem XIII.

TO inscribe the square (*M N O P*) in the pentagon (*A B C D E.*)

Extend the perpendicular A Q and B C till they cut each other in F; make the perpendicular F I and A H equal to half A F; then draw B H and B I, cutting A Q in L and K, so will L K be the side of the square sought. Fig. 14.

Problem XIV.

OF four given lines (*AB=6, BC=9, CD=8, AD=18*) the greatest being less than the sum of the rest, to make a quadrangle which may be inscribed in a circle.

In every quadrangle inscribed in a circle, the sum of the rectangles made of the sides, containing opposite angles, have the same proportion to each other as the diagonals, *i e.*, as, Fig. 15.

72 108 54 144
D C B+D A B to A B C+C D A, so is C A to D B
180 198 13+ 15+

Also,

144   54   72   108   162   48
ADC+ABC : BCD+DAB∷AD×BC+AB×CD : AC□
198   180   210 $190\frac{10}{11}$
The root is 13 [illegible]

Ag[illegible]

72   108   144   [illegible]   162   48
BCD+DAB : AD[illegible] AD×BC+AB×CD : BD□
180   [illegible]   210   231
The root is 15 +

From the last of these proportions find the diagonal B D; with which, and the other two sides, either B C and C D, or A B and A D, form a triangle, and about it draw the circle A B C D, and in it insert the other two sides of the quadrangle. *W. W. D.*

Problem XV.

T*O inscribe an heptagon, nonagon, or undecagon, &c.*

Fig. 16. Having obtained the side of a polygon, next bigger and next less, in the same circle, extend the diameter A B to D, and from C extend the said sides of the polygon next above or under to D and E, which distance bisect in F, and draw C F, the side of the polygon sought lies on that line C F, between C and the circumference.

The side of a septagon or heptagon, is nearly equal to the perpendicular of an equilateral triangle, whose side is radius, or an hexagon; so the side of a nonagon is nearly the perpendicular of an equilateral triangle, whose side is an octagon.

Problem XVI.

T*O describe a regular octagon on a given side (AB.)*

Fig. 17. On the middle of A B, erect the perpendicular C E; on the point C and distance A C, describe a semi-circle ADB; on

On the point D, and with the distance D A, draw A E B; so is E the center, and A E the radius of that circle, which contains the octagon, whose side is A B. *W. W. D.*

Problem XVII.

TO *describe a regular nonagon on a given line (A B.)*

Erect the perpendicular F C in the middle of A B; on B, with the distance A B, draw the arch A D, which bisect in E; on the point D, with D E, draw the arch E F; so is F the center, and A F the radius of that circle which contains the nonagon, whose side is A B. *W. W. D.* *Fig.* 18.

Problem XVIII.

TO *inscribe a regular nonagon in a given circle.*

On B, with the radius A B, draw D A C and D C, which extend to F; make E F = A B; on E and F draw E G and F G, then draw A G, which cuts the circle in H; so is D H the side of the nonagon sought. *W. W. D.* *Fig.* 19.

H B is the side of a regular polygon of 18 sides, and $\frac{3}{?}$ of the radius is the side of a regular nonagon.

Problem XIX.

TO *inscribe a regular undecagon in a circle.*

Divide A B in two at C; on A and C, with the distance A C; draw the arch C D I and A D; on the center I and distance I D, draw the arch D O; so is C O the side of the undecagon sought. *Fig.* 20.

Some say $\frac{9}{32}$ of the diameter is the side of an undecagon inscribed, or $\frac{1}{4}$ diameter, more $\frac{1}{8}$ of the said $\frac{1}{4}$.

Or,

Quarter a circle and inscribe an equilateral triangle, that part of the side which lies betwixt the diameter and the angle of the base is the side of a regular undecagon.

Problem XX.

TO describe a re[illegible]lar dodecagon on a given side (AB.)

Fig. 21. On the middle of AB, erect [illegible]e perpendic[illegible]lar CD; on A and B, with the distanc[illegible] AB, draw the arches AE and BE; on [illegible], with the [illegible]ance AE, draw the arch AD: so is D th[illegible] of the dodecag[illegible]n sought: For AEB is the angle [illegible]h[illegible]gon, and ADB is its half, the angle of a d[illegible]

Problem XXI.

ON a given line (AB) [illegible]o describe any polygon from 6 to 12 sides.

Fig. 22. Bisect AB in O; erect on [illegible] perpendicular OI; on B, with the distance AB, [illegible] the arch AC, which divide into 6 equal parts [illegible]rcle [illegible] and from the distance of each of those equal part[illegible] the arches DM, EN, FP, GQ, HR and AI; so is

C	center of a	Hexagon,	and	CA	the radius.
D		Heptagon,		DA	
E		Octagon,		EA	
F		Nonagon,		FA	
G		Decagon,		GA	
H		Endecagon,		HA	
I		Dodecagon,		IA	

Problem XXII.

TO cut two given lines (AB and AC) into four, so that all four shall be continually proportional, (CF : DF : : ED : BE.)

Fig. 23. Set AB and AC at right angles, and draw BC; bisect AB, and draw the semi-circle BDO through D; draw DF parallel to BO, and DE parallel to CO; so will it be,

BE : ED : : ED : DF
ED : DF : : DF : FC

Problem

Problem XXIII.

FRom a ftrait line given (I) to cut off a part (CD) which fhall be a mean betwixt the other part and another line given (H.)

Make the line CD = lines H and I given, on E, where the two lines meet, erect the perpendicular EF, and draw the femi-circle on the diameter CD: bifect CE=H in B; on B, with the radius BF, draw the arch FG; fo will DG : GE : EC ∺. Fig. 24.

Problem XXIV.

GIVEN the fum of the extremes (AB) and the mean (BD) to find the extremes (AF and FB) feverally.

On AB erect a femi-circle, and on B erect a perpendicular equal to BD; through D, parallel to AB, draw ED and EF parallel to DB; fo is AF one and FD the other extreme. *Q. E. I.* Fig. 25.

Problem XXV.

GIVEN the difference of the extremes (AB) and the mean (BC) to find the extremes (BE and BF) feverally

On AB, the difference, erect the perpendicular BC= mean. Bifect AB in D; on D, with the diftance CD, draw the femi-circle, whofe diameter FE contains the two extremes fought, *i. e.* BE and BF. Fig. 26.

Problem XXVI.

THE excefs of the diagonal above the fide of a fquare being given (AB) to find the fide (AD.)

Arithme-

Arithmetically thus:

Fig. 27. To the excess given A B, add the square root of double the square of the excess for the side A D sought.

Geometrically.

On B erect the perpendicular B C=A B, and draw A C D; on C, with the distance C B, draw the arch B D; so will A D be the [illegible] the square, and A E the diagonal, which exceeds A D by A B.

Problem XXVII.

GIVEN *the sum of the side and diagonal (A B) to find them separately (G A and G B.)*

Fig. 28. On A B draw the semicircle A C B; erect C D perpendicular. Bisect A D in E, [illegible], and the distance E C, draw C F, the chord C F is the diagonal sought, which set from B to G; so will G A be the side sought.

Problem XXVIII.

GIVEN *the area of a rectangled parallelogram, (36) and the proportion of the sides, as 4 to 1, to find the sides.*

Say 4 : 1 : : 36 : 9, whose square root is 3, the breadth; as 1 : 4 : : 36 : 144, whose square root is 12, the length.

Problem XXIX.

GIVEN *the difference of the sides and sum of the sides of a right-angled parallelogram, to find the sides.*

To the half sum add the half difference for the length; from the half sum take the half difference for the breadth.

Problem

Problem XXX.

GIVEN the difference of the sides, and of their squares, to find the sides of a rectangled parallelogram.

Divide the difference of the squares by the difference of the sides, the quotient is the sum.

Or,

Divide the difference of the squares by the sum of the sides, the quotient is the difference of their sides, by 6. 2 *Euclid.*

Problem XXXI.

IN any triangle, the sum of every two sides given, to find them severally.

From half the sum of all the given numbers, subtract each, and the remainders are the sides required, *i. e.* each particularly of the letter wanting.

Problem XXXII.

THE sides of a trapezium, (ABCD) and the diagonal (AC) given, to find the other (BD.)

□AC+BC—AB; divide half the remainder by AC, the quotient is CE. *Fig.* 29.

□AC+AD—CD; divide half the remainder by AC, the quotient is AF, by 47. 1. Find BE and DF, add CE and AF, and subduct the sum from AC, the remainder EF = DG, then □DG + □BG = □BD. *Q. E. I.*

Problem

Problem XXXIII.

IN a right-angled triangle (ABC,) right-angled at B, is given AC=13, and the Sum of AB and BC=17, to find AB and BC apart.

Fig. 30. Make the ſquare ED=17 for the ſide, and 289 will be the area; from whence take the ſquare of AC=169, the remainder 120=2 rectangles EB and BD, each of which is a mean betwixt the ſquare AB and BC, and double the triangle ABC.

Wherefore take half 17=8.5, whoſe ſquare is 72.25: from which take 60, and there reſt 12.25, whoſe ſquare root is 3.5; which taken from 8.5, leaves 5 for CB, and added to 8.5, make 12 for AB.

Like to this is the following. "In a right angled tri-"angle ABC, right-angled at B, is given BC=40, and "the ſum of AB and AC together 150, to find them ſe-"verally.

Square the perpendicular, and divide that ſum 1600 by double the ſum of AB and BC=300, the quotient is $5\frac{1}{3}$, which added to the half ſum of AB and BC=75, gives $80\frac{1}{3}$ for the hypothenuſe AC; and taken from it, gives $69\frac{2}{3}$ for AB.

Or,

From the ſquare of the ſum of AB and AC 150=22500 [illegible] the ſquare of BC 40=1600 and divide the remainder =20900 by double the ſum of AB and BC=300, the quotient is $69\frac{2}{3}$ for AB, &c.

Problem XXXIV.

IN the rectangle (ABCD) are given BE=16 and EF=2. Quære the area.

Fig. 31. Seeing the angles are right, 'tis as FE : ED :: ED : EA; and again, as ED : EA :: EA : EB. And ſeeing EB is the third number, and ED and EA are two means proportional betwixt 2 and 16, reduce each into their leaſt proportion, which is 1 and 8, and extract the cube root of 8 is 2, which doubled (becauſe 2.16 were halved before) give 4 for ED; then is EA 8, and conſequently the area 160.

Problem XXXV.

IN the triangle (ABC) are the three perpendiculars (AD=56, BF=60, and EC=$64\frac{8}{13}$) given, to find the three sides.

In the triangle, whose sides are 13, 14, 15, the perpendicular is found to be 12; wherefore say, as Fig. 32.

$$12 : 14 :: \begin{cases} 56 : 65\frac{1}{3}\ AB \\ 60 : 70\ \ AC \\ 64\frac{8}{13} : 75\frac{5}{13}\ BC \end{cases}$$

Problem XXXVI.

IN an isosceles triangle (ABC) are two circles touching one another, and the sides of the triangle; the diameter of one is 12 and the other 8. Quære the sides of the triangle.

EH is the difference of their radii =2, and EF the sum of the radii =10; then say, if 2 give 10, what will EG=6 give? Answer, 30 for AE; to which add ED=6, and it makes 36 for AD, the perpendicular; then □AE—□EG=□AG; then, as AG : AE :: AD : AC=AB; then □AC—□AD=□CD, which doubled, is BC the base. *Q. E. I.* Fig. 33.

Problem XXXVII.

THE hypothenuse AB=48, and the area of a right-angled triangle=384, given to find the sides AC and CB.

On AB=48, the hypothenuse, make a semi-circle, on whose center D, erect the perpendicular DG; then double the area given 768, which divide by the hypothenuse=48, the quotient is DE=16 of the isosceles triangle AEB; through E, parallel to AB, draw EC, then AC and BC are the sides sought. *Q. E. I.* Fig. 34.

□DC—CF□=DF□, then AD+DF=AF, and □AF+FC□=AC□, then AB—AF=FB, and □CF+FB□=BC□,

Problem XXXVIII.

THE *diameter of a given circle*=10, *in which make the greatest rectangle possible, whose length shall be double its breadth. Quære the sides.*

Fig. 35. Divide the square of the diameter by 5, the quotient is 20, whose square root is the breadth: Deduct 20 out of 100, remains 80, whose square root is the length. *Q. E. I.*

Problem XXXIX.

THE *diameter of a circle*=8, *in it to inscribe on the diameter an equilateral triangle, whose vertex shall touch the circumference. Quære its side.*

Fig. 36. Cross the circle with the two diameters, AB, EF; in it inscribe an equilateral triangle ACD, whose vertex shall be A, so will the diameter EF cut off the equilateral AHK. *W. W. D.*

To find the side.

□AG+$\frac{1}{3}$□AG=□AH.

Problem XL.

IN *the circle (AFBGCD) having the square (ABCD) inscribed, is given each segment's area* =224 *(ABF) contained betwixt the side of the square and circumference, to find the diameter.*

Fig. 37. Suppose the diameter =1, the area will be $\frac{11}{14}$, whereof $\frac{1}{4}$ will be $\frac{11}{56}$=AEBF, the area of the triangle AEB=$\frac{1}{8}$, which

which subtracted from $\frac{11}{56}$ leaves $\frac{1}{14}$ for ABF: then say, as $\frac{1}{14}$: □ 1 : : 224 quantity given : □ diameter $=3136$, whose square root is $56=$diameter AC, supposing the proportion of the diameter to the circumference, as 7 to 22.

Problem XLI.

In a circle, whose diameters (BD)=65 is a triangle inscribed (ABC) the side (AB=52 and (BC)=56. Quære the side (AC.)

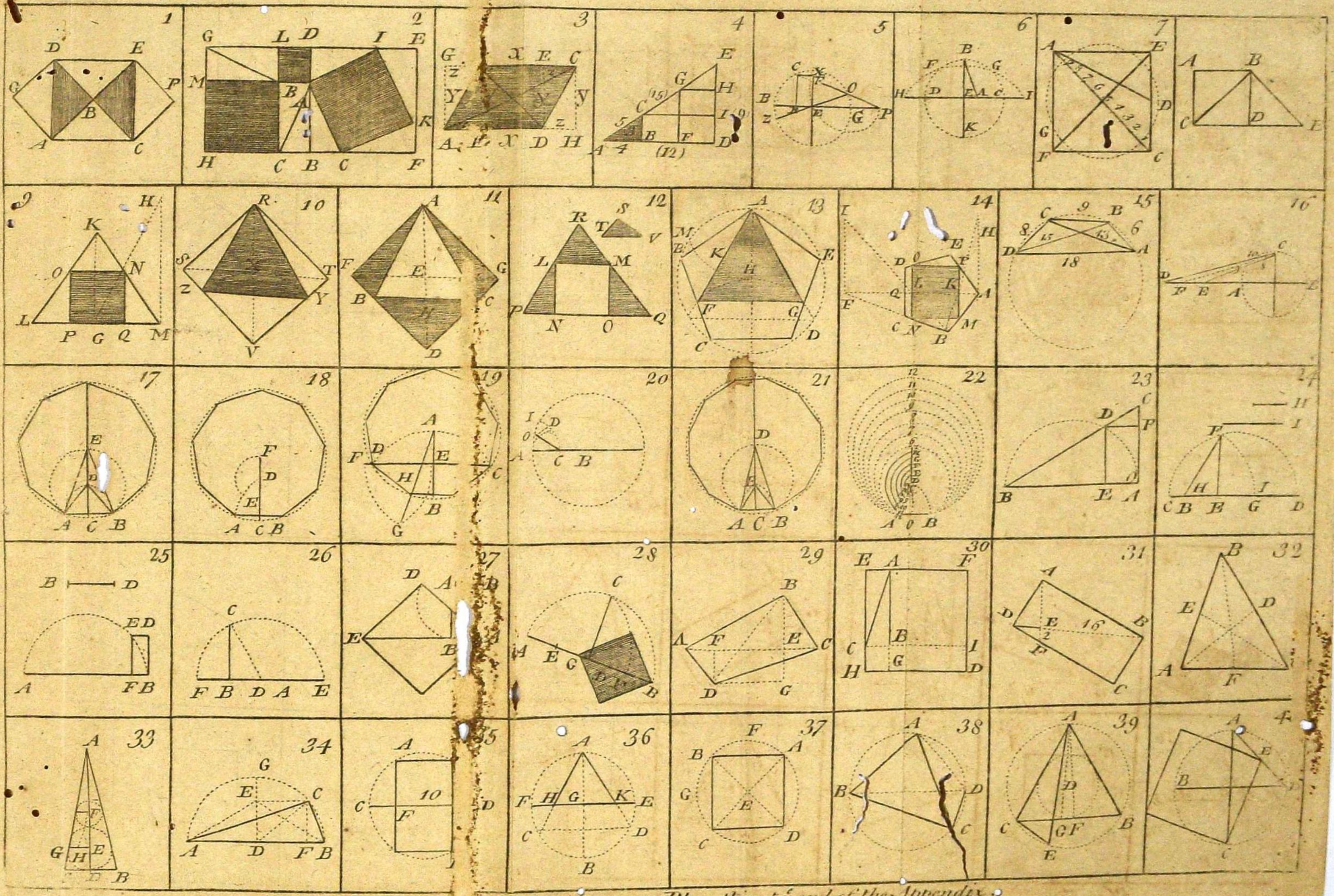
□BD — □AB = □AD and □BD — □BC = □CD, then multiply AD by BC, also AB by DC; these two products=3900 divide by BD=65, the quotient 60 =AC, *Q. E. I.* *Fig.* 38.

Problem XLII.

In the circle (ACEB) is inscribed a triangle (ABC) AB=13, BC=14, AC=15, and in that triangle is inscribed a circle. Quære the two diameters of both circles.

Find the perpendicular AF=12, then because the triangles ACE and ABF are similar, say AF : AB : : AC : AE, the diameter of the circumscribing circle $=16\frac{1}{4}$. Then seek the area of the triangle=84, which doubled is 168; then add all the three sides, they make 42, by which divide 168, the quotient is 4=GD, half the diameter of the inscribed circle. *Fig.* 39.

The



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